Differential Forms

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I. ONE-FORMS AND VECTOR FIELDS

A. One-forms and vector fields

1. A *one-form*, or a *differential one-form*, on an open interval in $\mathbb{R}^1$ is an expression of the form

$$\omega = f(x)\, dx$$

with $f : U \to \mathbb{R}^1$ being functions with continuous derivatives (C^∞ on U).

2. A one-form on an open subset in $\mathbb{R}^2$ is an expression of the form

$$\omega = f(x, y)\, dx + g(x, y)\, dy$$

with $f, g : U \to \mathbb{R}^2$ again being a function with continuous (this time partial) derivatives (C^∞ on U).

3. The sum and products of one-forms are also one-forms.

4. *Vector fields* on an open subset $U \subset \mathbb{R}^n$ is a function $\mathbf{F} : U \to \mathbb{R}^n$. A vector field is *smooth* if the component functions are smooth.

5. There is a correspondence between one-forms and vector fields. Given a one-form, there is an associated smooth vector field and vice-versa.

B. Exact one-forms and conservative vector fields

6. A *differential* of a function is a one-form on U.

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial x}dx$$

7. The vector field associated to the differential df of a function f is a vector field, namely ∇f.

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x}\right)$$

8. By setting f as x, we find that the placeholders dx, dy, dz are differentials of the component functions x, y, z.

9. Like not all vector fields can be written as the gradient of a function, not all one-forms are differentials of functions. Such one-forms and vector fields pairs are special, and hence have their own name: *exact one-forms.*

$$\omega = df$$

10. On the other hand, a vector field is called conservative if it is a gradient of a function.

$$\mathbf{F} = \nabla f$$

Here, f is called the *potential* of $\mathbf{F}$.

11. A "screening test" for checking whether a one-form is exact is checking if it is *closed*, that is,

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

The role of the expression is clear, as Clairaut-Schwarz theorem clearly states that partial derivatives commute if they are continuous. Simply stated, if a function is exact, it is closed.

12. A "screening test" for conservative vector fields that have continuously differentiable components would be the following

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$$

12. A one-form on an open subset in $\mathbb{R}^3$ is closed if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \qquad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \qquad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}$$

13. Analogically, a "screening test" for conservative vector fields that have continuously differentiable would be

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}, \qquad \frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x}, \qquad \frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y}$$

C. Changes of variables

14. For reparameterisation-invariance, we define one-form transformations under the change of variable x to t as

$$\eta = \left(f(x(t))\frac{dx}{dt}\right) dt$$

15. The *pullback of a function* is the smooth function defined as

$$\phi * f = f \circ \phi : V \to \mathbb{R}$$

and explicitly written, $\phi * f(t) = f(\phi(t))$. The term is justified by the fact that in the chain of maps $V \to^{\phi} U \to^{f} \mathbb{R}$, we "pull back" the function to a function from V to $\mathbb{R}$.

16. The *pullback of an one-form* is the one-form defined as

$$\phi * \omega = \left(f(\phi(t)) \frac{d\phi}{dt} \right) dt = \left(\phi * f(t) \frac{d\phi}{dt} \right) dt$$

D. The pullback of a one-form

17. The definition above can be further generalised in cases where the function ϕ is multivariate vector function (that is, $V, U \in \{1, 2, 3\}$) like the following

$$\phi * \omega = \left(f(\phi(t)) \frac{d\phi}{dt} \right) dt = \left(\phi * f(t) \frac{d\phi}{dt} \right) dt$$

and explicitly written, $\phi * f(\mathbf{t}) = f(\phi(\mathbf{t}))$.

18. Important properties of one-forms were

    1. If ω and η are one-forms, then $\omega + \eta$ is a one-form.

    2. If ω is a one-form and f is a smooth function, then $f\omega$ is a one-form.

    3. An exact one-form is one that can be written as a differential of a function: $\omega = df$

19. We then define the pullback axiomatically to be consistent with these properties like the following

    1. $\phi * (\omega + \eta) = \phi * \omega + \phi * \eta$

    2. $\phi * (f\omega) = (\phi * f)(\phi * \omega)$

    3. $\phi * (df) = d(\phi * f)$

20. These definitions are sufficient to determine the pullback of any one-form. The *pull back of dx* is then

$$\phi * (dx) = \sum_{i=1}^{m} \frac{\partial x}{\partial t_i} dt_i$$

for $m = 3$,

$$\phi * (dx) = \frac{\partial x}{\partial t_1} dt_1 + \frac{\partial x}{\partial t_2} dt_2 + \frac{\partial x}{\partial t_3} dt_3$$

21. The *pullback of an one-form* is then generalised as

$$\phi * \omega = f(\phi(\mathbf{t})) \sum_{i=1}^{m} \frac{\partial x}{\partial t_i} dt_i + g(\phi(\mathbf{t})) \sum_{i=1}^{m} \frac{\partial y}{\partial t_i} dt_i + h(\phi(\mathbf{t})) \sum_{i=1}^{m} \frac{\partial z}{\partial t_i} dt_i$$

II. INTEGRATING ONE-FORMS: LINE INTEGRALS

A. Integrating a one-form over an interval

22. The *integral of a one-form* over $[a, b]$, $a \leq b$ is defined as

$$\int_{[a,b]} \omega = \int_a^b f(x)\, dx$$

23. The *orientation of an interval* is a choice of direction. It can either be one of increasing numbers ($[a, b]_+$) or decreasing numbers ($[a, b]_-$). We define the *canonical orientation* to be the orientation of increasing real numbers.

24. The *integral of a one-form over the oriented interval* $[a, b]_\pm$ is defined as the following

$$\int_{[a,b]_\pm} \omega = \pm \int_a^b f(x)\, dx$$

25. Integrals of one-forms over intervals are invariant under orientation-preserving reparametrisations. Given $\phi(c) = a$ and $\phi(d) = b$,

$$\int_{[c,d]} \phi * \omega = \int_{[a,b]} \omega$$

Explicitly,

$$\int_c^d f(\phi(t)) \frac{d\phi}{dt} dt = \int_a^b f(x)\, dx$$

The substitution formula for definite integrals is simply the statement that integrals of one-forms over intervals are invariant under pullback.

26. Integrals of one-forms over intervals pick a sign under orientation-reversing reparametrisations.

$$\int_{[c,d]} \phi * \omega = \int_{[a,b]_-} \omega = - \int_{[a,b]} \omega$$

B. Parametric curves in $\mathbb{R}$

27. *Parametric curves*

$$\alpha : [a, b] \to \mathbb{R}^n$$

$$t \mapsto \alpha(t) = (x_1(t), ..., x_n(t))$$

28. *Closed parametric curves* are curves that don't have endpoints.

29. The set $\partial C = \{\alpha(a), \alpha(b)\}$ that consists of the endpoints of C is called the *boundary of the curve.*

30. The *tangent vector* or *velocity vector* to a parametric curve

$$\mathbf{T} : [a, b] \to \mathbb{R}^n$$

$$t \mapsto \mathbf{T}(t) = \alpha'(t) = (x_1'(t), ..., x_n'(t))$$

31. *Orientation of a curve* is given by the choice of direction of the curve.

32. *Parametric curves are orientated* where the direction is given by the direction of the tangent vector at each point on the curve.

33. *Reparametrisations of a curve* can be done through pullbacks, where the following pullback is another parametrisation of the same curve

$$\phi * \alpha : [c, d] \to \mathbb{R}^n$$

$$u \mapsto (\phi * x_1(u), ..., \phi * x_n(u)) = (x_1(\phi(u)), ..., x_n(\phi(u)))$$

To add, as $d\phi/du$ is continuous and never zero (by definition) on $[c, d]$, it is everywhere positive or everywhere negative.

34. *Orientation-preserving reparametrisations* are cases in which $d\phi/du > 0$ for all points.

35. *Orientation-reversing reparametrisations* are cases in which $d\phi/du < 0$ for all points.

36. If a piecewise parametric curve is the union of a number of parametric curves, and each paramteric curve is smooth, we call the piecewise paramteric curve *piecewise smooth.* Additionally, if the curve doesn't intersect itself, we call the curve *simple.*

C. Line integrals

37. *The pull back of a one-form on an open subset in* $\mathbb{R}^2$

$$\alpha * \omega = \left(f(\alpha(t)) \frac{dx}{dt} + g(\alpha(t)) \frac{dy}{dt} \right) dt$$

38. *The pull back of a one-form on an open subset in* $\mathbb{R}^3$

$$\alpha * \omega = \left(f(\alpha(t)) \frac{dx}{dt} + g(\alpha(t)) \frac{dy}{dx} + h(\alpha(t)) \frac{dz}{dt} \right) dt$$

39. We can translate the language of differential forms into the language of vector fields. If $\mathbf{F}$ is the associated vector field to ω,

$$\alpha * \omega = (\mathbf{F}(\alpha(t)) \cdot \mathbf{T}(t))\, dt$$

38. *Oriented line integral of a one-form along* α

$$\int_\alpha \omega = \int_{[a,b]} \alpha * \omega$$

$$\int_\alpha \omega = \int_a^b \left(f(\alpha(t)) \frac{dx}{dt} + g(\alpha(t)) \frac{dy}{dt} \right) dt$$

40. *Line integrals over piecewise parametric curves* can be done simply by adding up the integrals of the individual curves.

41. *Line integrals are invariant under orientation-preserving reparametrisations.*

    (i) If ϕ preserves orientation,

$$\int_\alpha \omega = \int_{\phi * \alpha} \omega$$

    (ii) If ϕ reverses orientation,

$$\int_\alpha \omega = -\int_{\phi * \alpha} \omega$$

D. Fundamental theorem of line integrals

42. *The fundamental theorem of line integrals.* Let $\omega = df$ be an exact one-form.

$$\int_\alpha \omega = \int_\alpha df = f(\alpha(b)) - f(\alpha(a))$$

43. *The line integrals of an exact one-form* along two curves that start and end at the same points are equal.

44. *The line integral of an exact one-form* along a closed curve vanishes.

45. *The fundamental theorem of line integrals for vector fields.*

$$\int_a^b \mathbf{F}(\alpha(t)) \cdot \mathbf{T}(t) dt = \int_a^b \nabla f(\alpha(t)) \cdot \mathbf{T}(t) dt$$
$$= f(\alpha(b)) - f(\alpha(a))$$

$$\int_c \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(a)) - f(\mathbf{r}(b))$$

E. Poincare's lemma for one-forms

46. *Poincare's lemma, version I.* ω is exact if and only if ω is closed. In the language of vector calculus, $\mathbf{F}$ is

conservative if and only if it is curl-free:

$$\nabla \times \mathbf{F} = 0$$

47. *Equivalent formulations of exactness on* $\mathbb{R}^n$

   1. ω is exact ($\mathbf{F}$ is conservative).
   2. ω is closed ($\mathbf{F}$ passes the screening test).
   3. The integral $\int_\alpha \omega = 0$ for any closed parametric curve α.
   4. Line integrals of ω are path independent.

48. *Poincare's lemma, version II.* Let ω be a one-form defined on an open subset $U \subset \mathbb{R}^n$ that is simply connected. Then ω is exact if and only if it is closed.

III. k-FORMS

A. Differential forms revisited: an algebraic approach

49. *The basic one-form* dx_i is a linear map which takes a vector and projects it onto the x_i-axis.

$$dx_i(u_1, ..., u_n) = u_i$$

50. The rigorous meaning of these placeholders allow us to write a general linear map $M : \mathbb{R}^3 \to \mathbb{R}$ as

$$M = A\;dx + B\;dy + C\;dz$$

where A, B, C are just constants. In other words, it is an arbitrary linear combination of the three projection operators. In general, given an abstract vector space V, the set of linear maps $M : V \to \mathbb{R}$ forms a vector space itself, which is called the "dual vector space" and denoted by V^*.

51. For any point $P \in U$, the one-form ω defines a linear map $\mathbb{R}^3 \to \mathbb{R}$ (or equivalently an element of the vector space dual to $\mathbb{R}^3$). This is the dual concept to vector fields: a vector field is a rule that assigns to all points on U a vector in $\mathbb{R}^3$, while a one-form is a rule that assigns to all points on U a linear map $\mathbb{R}^3 \to \mathbb{R}$.

52. *The basic two-form* $dx_i \wedge dx_j$ is a multilinear map which takes two vectors and maps them into the following determinant.

$$dx_i \wedge dx_j(\mathbf{u}, \mathbf{v}) = \det \begin{pmatrix} u_i & v_i \\ u_j & v_j \end{pmatrix}$$

53. *The basic three-form* $dx_i \wedge dx_j \wedge dx_k$ is a multilinear map which takes three vectors and maps them into the following determinant.

$$dx_i \wedge dx_j \wedge dx_k(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \det \begin{pmatrix} u_i & v_i & w_i \\ u_j & v_j & w_j \\ u_k & v_k & w_k \end{pmatrix}$$

54. *The basic k-form* is a multilinear map which takes k-vectors and maps them into the following determinant.

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k}(\mathbf{u}^1, ..., \mathbf{u}^k) = \det \begin{pmatrix} u^1_{i_1} & \dots & u^k_{i_1} \\ \vdots & \ddots & \vdots \\ u^1_{i_k} & \dots & u^k_{i_k} \end{pmatrix}$$

55. *Antisymmetry of basic k-forms.* The basic two-forms satisfy the following properties.

$$dx_i \wedge dx_j = -dx_j \wedge dx_i$$

In particular,

$$dx_i \wedge dx_i = 0$$

Thus, basic k-forms in $\mathbb{R}^n$ are for $1 \leq k \leq n$.

55. The basic k-forms can be given a geometric interpretation.

   1. The basic three-form calculates the oriented volume of the parallelepiped spanned by the three vectors that are taken as input.
   2. The basic two forms calculate the oriented area of the project of the parallelogram spanned by the two vectors that are taken as input.

56. *k-forms in* $\mathbb{R}^3$. Let $U \subset \mathbb{R}^3$ be an open subset.

   1. A zero-form is a smooth function $f : U \to \mathbb{R}$
   2. A one-form is an expression of the form

$$f\;dx + g\;dy + h\;dz$$

   for smooth functions $f, g, h : U \to \mathbb{R}$.

   3. A two-form is an expression of the form

$$f\;dy \wedge dz + g\;dz \wedge dx + h\;dx \wedge dy$$

   for smooth functions $f, g, h : U \to \mathbb{R}$.

   4. A three-form is an expression of the form

$$f\;dx \wedge dy \wedge dz$$

   for a smooth function $f : U \to \mathbb{R}$.

57. We thus can give a geometric meaning to k-forms. A k-form assigns a multilinear map $(\mathbb{R}^3)^k \to \mathbb{R}$ to all points in U. In other words, a k-form assigns a notion of a k-dimensional oriented volume for the corresponding projection of the k-dimensional parallelepiped generated by the k vectors.

58. *Correspondence between one-forms and vector fields.*

differential form	vector calculus
f	f
$f\ dx + g\ dy + h\ dz$	(f, g, h)
$f\ dy \wedge dz + g\ dz \wedge dx + h\ dx \wedge dy$	(f, g, h)
$f\ dx \wedge dy \wedge dz$	f

B. Multiplying k-forms: the wedge product

59. *The wedge product* $\omega \wedge \eta$ is a $(k+m)$-form defined by

$$\omega \wedge \eta = fg\ dx_{i_1} \wedge \ldots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_m}$$

60. *Comparing* $\omega \wedge \eta$ *to* $\eta \wedge \omega$

$$\omega \wedge \eta = (-1)^{km} \eta \wedge \omega.$$

61. The wedge product of two one-forms is the cross product of the associated vector fields.

62. The wedge product of a one-form and a two-form is the dot product of the associated vector fields.

C. Differentiating k-forms: the exterior derivative

63. *The exterior derivative of a 0-form* f on $U \subset \mathbb{R}^n$ is the one form df on U given by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

64. *The exterior derivative of a* k*-form* ω on $U \subset \mathbb{R}^n$ is the one form $d\omega$ on U given by

$$d\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} d(f_{i_1 \cdots i_k}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

65. *The exterior derivative in* $\mathbb{R}^3$

1. If f is a zero-form on $U \subset \mathbb{R}^3$, then its exterior derivative df is the one-form

$$df = \frac{\partial f}{\partial x}\ dx + \frac{\partial f}{\partial y}\ dy + \frac{\partial f}{\partial z}\ dz.$$

2. If ω is a one-form on $U \subset \mathbb{R}^3$, then its exterior derivative $d\omega$ is the one-form

$$d\omega = d(f) \wedge dx + d(g) \wedge dy + d(h) \wedge dz$$

$$= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dy \wedge dz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) dz \wedge dx$$

$$+ \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy.$$

3. If ω is a two-form on $U \subset \mathbb{R}^3$, then its exterior derivative $d\omega$ is the one-form

$$d\eta = d(f) \wedge dy \wedge dz + d(g) \wedge\ dz \wedge dx$$

$$+ d(h) \wedge dx \wedge dy.$$

$$= \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) dx \wedge dy \wedge dz.$$

66. Exterior derivatives are linear, that is,

$$d(a\omega + b\eta) = ad\omega + bd\eta$$

67. *The graded product rule for the exterior derivative.*

$$d(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^k \omega \wedge d(\eta)$$

68. The four non-vanishing cases of the graded product rule

1. two one-forms

$$d(fg) = (df)g + f(dg)$$

2. a zero-form and a one-form

$$d(f\eta) = (df) \wedge \eta + f(d\eta)$$

3. a zero-form and a two-form

$$d(f\eta) = (df) \wedge \eta + f(d\eta)$$

4. a one-form and a one-form

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta - \omega \wedge (d\eta)$$

69. $d^2 = 0$

D. The exterior derivative and vector calculus

70. *The gradient of a function* is the vector field associated to the exterior derivative of a zero-form.

71. *The curl of a vector field* is the vector field associated to the exterior derivative of a one-form.

72. *The divergence of a vector field* is the exterior derivative of a two-form.

73. Vector calculus identities, part I

1. $\nabla(fg)$
2. $\nabla \times (f\mathbf{F})$
3. $\nabla \cdot (f\mathbf{F})$
4. $\nabla \cdot (\mathbf{F} \times \mathbf{G})$

74. Vector calculus identities, part II

1. $\nabla \times (\nabla f) = 0$
2. $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

75. Vector calculus identities, part III

$$\nabla \cdot (f(\nabla g \times \nabla h)) = \nabla f \cdot (\nabla g \times \nabla h)$$

76. Vector calculus identities, part IV

1. $\nabla(\mathbf{F} \cdot \mathbf{G})$
2. $\nabla \times (\mathbf{F} \times \mathbf{G})$

Here, $(\mathbf{G} \cdot \nabla)\mathbf{F}$ means

$$(\mathbf{G} \cdot \nabla)\mathbf{F} = G_1 \frac{\partial}{\partial x}\mathbf{F} + G_2 \frac{\partial}{\partial y}\mathbf{F} + G_3 \frac{\partial}{\partial z}\mathbf{F}$$

E. Physical interpretation of grad, curl, and div

F. Exact and closed k-forms

77. We say ω is *closed* if $d\omega = 0$ and we say ω is exact if there exists a $(k-1)$-form η such that $\omega = d\eta$.

78. If a k-form is exact, it is closed.

79. *Poincare's lemma for k-forms, version I.* ω on $\mathbb{R}^n$ is exact if and only if ω is closed.

80. *Poincare's lemma for k-forms, version II.* ω on open ball $U \subset \mathbb{R}^n$ is exact if and only if ω is closed.

[1] V. Bouchard. MATH 215: Calculus IV. `https://sites.ualberta.ca/~vbouchar/MATH215/front.html`.