General Relativity

Dane Jeon

Sogang University,
35 Baekbeom-ro, Mapo-gu, Seoul 04107, Republic of Korea

E-mail: danetrouveur@gmail.com

Abstract: This article contains notes I've taken for the general relativity course provided by Sogang university. The course was taught by professor Jeong-Hyuck park. As a novice to the field general relativity, I would welcome referrals to any errors. Kindly refer to by Email in such events.

Contents

1 Lecture 1

In this lecture, we start with an introduction about what general relativity is introducing two central equations used in the theory. We then move onto special relativity, deducing from Maxwell's equations that, according to his theory of electromagnetism, the speed of light is constant. We finish by deriving a particular transformation that can be derived from this fact (a special case of a Lorentz transformation), where we convert experienced times between an observer that is moving relative to another observer at rest.

1.1 Introduction

rmk. This course first begins with an introduction to special relativity which is the no gravity limit of general relativity. Then, we move on to the mathematics behind tensors and get a grasp of differential geometry, which is the language of the subject we're studying. Next, we learn the Einstein field equations, which relates information of curvature to the energy-momentum tensor.

$$\underbrace{G_{\mu\nu}}_{\text{information on curvature}} = 8\pi G \underbrace{T_{\mu\nu}}_{\text{energy-momentum tensor}}$$

Finally, we finish the course by looking at a few applications including (1) Newtonian gravity (which should be a limiting case of general relativity), (2) Schwarzschild geometry which can be seen as an exact solution of the Einstein field equations with spherical symmetry, (3) black-holes, (4) cosmology, and (5) gravitational waves.

rmk. The totality of general relativity can be summed up into the following two phrases by the physicist John Wheeler: "matter tells spacetime how to curve" and "spacetime tells matter/particulars how to move". The first part can be seen as a description of the Einstein field equations, whereas the second part can be seen as a description of the geodesic equation that we'll learn later on.

$$\ddot{x}^{\lambda} + \gamma^{\lambda}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0$$

1.2 The Speed of Light & Lorentz Transformations

def. The Maxwell equations in vacuum be summerised like the following

$$\begin{cases} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \mu_0 \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \end{cases}$$

thm. We can easily prove, through evaluating the curl of the curl of the electric field and the magnetic field, that the magnetic field and electric field both satisfy the same partial differential equations which is also a wave equation whose solution describe waves that travel in the speed of light.

lem. The curl of the curl of a vector field in $\mathbb{R}^3$ ($\nabla \times (\nabla \times \mathbf{F})$) can be calculated like the following.

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{F}) &= \varepsilon_{ijk} \mathbf{e}_i \partial_j (\nabla \times \mathbf{F})_k \\ &= \varepsilon_{ijk} \mathbf{e}_i \partial_j (\varepsilon_{klm} \partial_l F_m) \\ &= \varepsilon_{ijk} \varepsilon_{klm} \mathbf{e}_i \partial_j \partial_l F_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \mathbf{e}_i \partial_j \partial_l F_m \\ &= \mathbf{e}_i \partial_i \partial_j F_j - \mathbf{e}_i \partial_l \partial_l F_i \\ &= \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F} \end{aligned}$$

Few notes on the notation used. Note that all lower indices are summed over (alike the Einstein summation convention) and that we used the notation ∇^2 to denote the Laplacian of $\mathbf{F}$.

pf.

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) &= \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t} \right) \\ &= -\frac{\partial}{\partial t} (\nabla \times \mathbf{B}) \\ &= -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \end{aligned}$$

Combining this result with the lemma above and Gauss's law in vaccum, we obtain

$$\left(-\mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} + \nabla^2 \right) \mathbf{E} = 0$$

or, equivalently, using the wave operator/d'Alembertian $\Box$,

$$\Box\mathbf{E} = 0$$

For the magnetic field,

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{B}) &= \nabla \times \left(\mu_0\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) \\ &= \mu_0\varepsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}) \\ &= -\mu_0\varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}\end{aligned}$$

combining this result with the lemma above and the curl of the magnetic field, we finally obtain

$$\Box\mathbf{B} = 0$$

In vector notation,

$$\left(-\frac{\partial^2}{c^2\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\begin{pmatrix}\mathbf{E} \\ \mathbf{B}\end{pmatrix} = \mathbf{0}$$

cor. This second-order partial differential equation in one-dimensional case can be largely simplified through a light-cone coordinate system, which is described like the following

$$\begin{cases} x^+ &= x + ct \\ x^- &= x - ct \end{cases}$$

Then, the differential operators for time and the coordinate x can be rewritten in the new coordinate system via the chain rule as

$$
\begin{cases}
\dfrac{\partial}{\partial t} &= \dfrac{\partial x^+}{\partial t}\dfrac{\partial}{\partial x^+} + \dfrac{\partial x^-}{\partial t}\dfrac{\partial}{\partial x^-} = c(\partial_+ - \partial_-) \\[2ex]
\dfrac{\partial}{\partial x} &= \dfrac{\partial x^+}{\partial x}\partial_+ + \dfrac{\partial x^-}{\partial x}\partial_- = \partial_+ + \partial_-
\end{cases}
$$

In this new coordinate system, the equation above becomes

$$0 = \partial_+ \partial_- \Phi$$

which can be solved as

$$
\begin{aligned}
\partial_- \Phi &= f(x_-) \\
\Phi &= \int dx^- f(x^-) + c(x^+) \\
\Phi &= f_L(x + ct) + f_R(x - ct)
\end{aligned}
$$

In $\mathbb{R}^3$, the equation becomes

$$0 = \left(-\frac{\partial^2}{c^2 \partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Phi$$

whose general solution is

$$\Phi = \int C_{\boldsymbol{k}} \exp\{i(c|\boldsymbol{k}|t + \boldsymbol{k} \cdot \boldsymbol{x})\} + D_{\boldsymbol{k}} \exp\{i(-c|\boldsymbol{k}|t + \boldsymbol{k} \cdot \boldsymbol{x})\}\; d\boldsymbol{k}^3$$

Notice how the solution wave has a constant speed of c, which is the pinpoint of Einstein's theory of special relativity.

thm. We use a imaginary setup involving a train to derive the Lorentz transformation in a special case where one observer is on the train (S) and one is at rest (S'). As the speed of light must be equal for both observers,

$$
\begin{aligned}
c^2 \Delta t'^2 &= 4L^2 + v^2 \Delta t'^2 \\
(c^2 - v^2)\Delta t'^2 &= 4L^2 \\
\Delta t' &= \frac{2L}{\sqrt{c^2 - v^2}} = \frac{c\Delta t}{c^2 - v^2} = \frac{\Delta t}{\sqrt{1 - v^2/c^2}}
\end{aligned}
$$

On this note, an event can be considered a point within spacetime indepedent of the coordinate system used, and thus an event on the train can be denoted as:

$$
\begin{aligned}
\text{event} &\equiv (ct, x, y, z)_{\text{observer}} \\
&\equiv (ct', vt' + x_0, y_0, z_0)_{\text{observer}'}
\end{aligned}
$$

2 Lecture 2

In this section, we generalize the notion of a Lorentz transformations and learn about the space-time metric, a tool that can be used to express quantities that are invariant through Lorentz transformations.

2.1 General Lorentz Transformations

thm. To obtain general Lorentz transformations, consider the example of a observer on a train that travels along the x-axis being shot by a beam of light at B from an arbitrary point A. We first express the two events A and B in the coordinates of an observer outside and an observer inside.

$$\begin{cases} A : (t, x, y)_i = (t', x', y')_o \\ B : (\sqrt{1-(v/c)^2}t'', 0, 0)_i = (t'', vt'', 0)_o \end{cases}$$

The distance difference between the events can be written by the following two ways as

$$c^2(t'' - t')^2 = (x' - vt'')^2 + y^2.$$

Solving this quadratic equation with respect to t'',

$$t'' = \frac{t' - vx'/c^2 \pm 1/c\sqrt{(x' - vt) + (1 - v^2/c^2)y'^2}}{1 - v^2/c^2}$$

we choose the positive sign for the square-root term as we want the final time to be later than the initial time for the observer outside.

From this, we can also construct the same equivalence in terms of the observer inside, giving

$$c^2(\sqrt{1 - v^2/c^2}t'' - t)^2 = x^2 + y^2$$

Substituting for t'', and imposing that the equality holds for arbitrary y, we conclude that

$$\begin{cases} t = \dfrac{t' - vx'/c^2}{\sqrt{1 - v^2/c^2}} \\ \\ x = \dfrac{x' - vt'}{\sqrt{1 - v^2/c^2}} \end{cases}$$

Where the primed coordinates describe the coordinates for the observer outside and the unprimed coordinates describe the coordinates for the observer inside.

This general form of the Lorentz transform can be written in an alternate notation where $x^\mu = (x^0 = ct, x, y, z)$,

$$\begin{cases} x^0 &= \gamma(x'^0 - \beta x'^1) \\ x^1 &= \gamma(x'^1 - v x'^0) \\ x^2 &= x'^2 \\ x^3 &= x'^3 \end{cases}$$

In the case for a transformation from the moving frame to the rest frame, there would be a sign change,

$$\begin{cases} x'^0 &= \gamma(x^0 + \beta x^1) + C^0 \\ x'^1 &= \gamma(x^1 + \beta x^0) + C^1 \\ x'^2 &= x^2 + C^2 \\ x'^3 &= x^3 + C^3 \end{cases}$$

These transformations are called Lorentz transformations. If you would allow constant translations (like the second set of equations above), all the transformations would amalgamate to be a larger set, called Poincaré transformations. Considering these transformations in the light-cone coordinate system that we considered above, we obtain

$$\begin{cases} x'^+ = \gamma(x^+ + \beta x^+) = \gamma(1+\beta)x^+ = \sqrt{\dfrac{1+\beta}{1-\beta}}x^+ \\ \\ x'^- = \gamma(x^- - \beta x^-) = \gamma(1-\beta)x^- \sqrt{\dfrac{1-\beta}{1+\beta}}x^- \end{cases}$$

HW. Note that the following expression of distance is an invariant quantity:

$$(\Delta x^0)^2 - \Delta\vec{x}\cdot\vec{x} = (\Delta x'^0)^2 - \Delta\vec{x}'\cdot\vec{x}'$$

as

$$\begin{aligned} RHS = & \gamma^2(\Delta x^0 + \beta\Delta x^1)^2 - \gamma^2(\Delta x^1 + \beta\Delta x^0)^2 \\ & - (\Delta x^2)^2 - (\Delta x^3)^2 \\ = & (\Delta x^0)^2(\gamma^2 - \gamma^2\beta^2) + (\Delta x^1)^2(\gamma^2\beta^2 - \gamma^2) \\ & - (\Delta x^2)^2 - (\Delta x^3)^2 \\ = & (\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2 \\ = & LHS \end{aligned}$$

def. At this point, we introduce the spacetime metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. Using this metric, also called the Minkowskian, we can express this invariant quantity (Δs), known as the proper distance, as

$$\Delta s^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu$$

As infinitesimals,

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$$

def. Using this definition, we can can state that Lorentz transformations are the group of linear maps that leave the proper distance invariant.

$$\begin{aligned}&\Delta s'^2 = \eta_{\mu\nu}\Delta x'^\mu \Delta x'^\nu = \eta_{\mu\nu}L^\mu_\rho L^\nu_\sigma \Delta x^\rho \Delta x^\sigma = \\ &\Delta s^2 = \eta_{\rho\sigma}\Delta x^\rho \Delta x^\sigma.\end{aligned}$$

Notice that for the proper distance to be invariant, we require

$$\eta_{\rho\sigma} = \eta_{\mu\nu}L^\mu_\rho L^\nu_\sigma$$

In matrix notation, the above identity can be written as

$$\eta = L^T \eta L$$

To emphasize again, any Lorentz transformation would satisfy the matrix identity above. It is good to note that Lorentz transformations can be taken as the set of all boosts and spacial rotations.

2.2 Representation theory of the Lorentz Group

rmk. A simple corollary of the identity above is that the set containing all transformations would be closed under multiplication and inverses as,

$$\begin{aligned}(L_1L_2)^T\eta(L_1L_2) &= \eta \\ (L^{-1})^T\eta L^{-1} &= \eta\end{aligned}$$

Here, we recall the definition of a group and a lie group.

recall. A group is a set equiped with a binary operation that is associative, contains an identity element within the group, and has an inverse element for every element such that it multiplies to give the identity element above. Some examples of a groups are the orthogonal groups $O(n)$ where n denotes the order of the group. They are the collection of all $n \times n$ matrices such their matrix multiplication with their transpose gives the identity matrix.

$$O(n) = \{A \in \mathcal{M}_{n\times n}(\mathbb{R}) \mid A^T A = I_n\}$$

def. The subgroup of the orthogonal group whose elements additionally have a unit determinant is called the special orthogonal group.

$$SO(n) = \{A \in O(n) \mid \det(A) = 1\}$$

def. The group of all invertible $(p+q) \times (p+q)$ matrices satisfying

$$A^T \eta A$$

for $\eta = \text{diag}(\underbrace{-1, ... -1}_{p-\text{times}}, \underbrace{1, ...1}_{q-\text{times}})$ is called the indefinite-orthogonal group. The subgroup, called the special indefinite-orthogonal group is, analogous to the case above, members of $O(p, q)$ with a unit determinant. In this way, the Lorentz group can be reframed as the special indefinite-orthogonal group $SO(1, 3)$.

def. A lie group is a group that is also a differentiable manifold such that the group multiplication map and the inverse map are differentiable.

3 Lecture 3

In this lecture we expand the discussion of groups that we had a prelude on in the last lecture, and future see the similarity between the Lorentz group that satisfies a certain identity and the rotational group that satisfies another.

3.1 The Lie Algebra of the Matrix Lorentz Group

recall. We know that the two-dimensional Lorentz transformation for time and space is given as

$$t' = \gamma(t + \beta x)$$
$$x' = \gamma(x + \beta t)$$

In matrix form,

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} = \begin{pmatrix} t \\ x \end{pmatrix}$$

Here gamma is given as $\gamma = 1/\sqrt{1-\beta^2}$, where $\beta = v/c$, satisfies $-1 < \beta < 1$, and also is a continuous parameter for the specific transformation.

cor. The fact that $\gamma^2 - (\gamma\beta)^2 = 1$ allows us to take

$$\gamma = \cosh\phi$$
$$\gamma\beta = \sinh\phi$$
$$\beta = \tanh\phi$$

Letting us to write the transformation via hyperbolic functions,

$$L(\phi) = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}$$

Note that the matrix satisfies

$$L^t \eta L = \eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

cor. The derivative of the matrix is–

$$\frac{dL(\phi)}{d\phi} = \begin{pmatrix} \sinh\phi & \cosh\phi \\ \cosh\phi & \sinh\phi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} L(\phi)$$

Considering the higher derivatives,

$$\frac{d^n L}{d\phi} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^n L(\phi)$$

In this manner, we can obtain an exponential expansion of the Lorentz transformation like the following (also note that $L(\phi = 0) = I$)

$$\begin{aligned} L(\phi) = \sum_{n=0}^{\infty} \frac{\phi^n}{n!} \frac{d^n L}{d\phi^n}\bigg|_{\phi=0} &= \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} 0 & \phi \\ \phi & 0 \end{pmatrix}^n \\ &= \exp\{\phi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\} \\ &\approx I + \phi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

Notice that we can thus approximate

$$\left(I + \phi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)^T \eta \left(I + \phi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \approx \eta$$

thm. From above, we can deduce that η satisfies

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^T \eta + \eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0$$

we denote the group of all matrices η that satisfy this as $\mathrm{SO}(1,1)$.

rmk. Notice that the Pauli matrices satisfy this, where

$$\sigma^x \sigma^z + \sigma^z \sigma^x = 0$$

thm. A great analogous situation is rotation on a plane, where

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \exp\{\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}\} \begin{pmatrix} x \\ y \end{pmatrix}$$

and where the matrix I satisfies

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^T I + I \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0$$

We denote all the matrices that satisfy this equation as SO(2).

thm. Like the examples above, the Lorentz transformations can be seen as a $\mathrm{SO}(1,3)$, denote-able in a general form as $L = e^M \approx I + M$. It should satisfy

$$L^T \eta L = \eta = \mathrm{diag}(-1, 1, 1, 1)$$

being a Lorentz transformation the following approximation can be made.

$$(I + M)^t \eta (I + M) \approx \eta$$

in an expanded form, we finally arrive at the key identity

$$M^t \eta + \eta M = 0$$

prop. We can use the identity we have derived above to give an explicit form for the Lorentz transformation's exponential M.

$$(\eta M)^t = -\eta M$$

and from the above,

$$\begin{pmatrix} 0 & \phi_x & \phi_y & \phi_z \\ \phi_x & 0 & \theta_z & -\theta_y \\ \phi_y & -\theta_z & 0 & \theta_x \\ \phi_z & \theta_y & -\theta_x & 0 \end{pmatrix}$$

We state without explaination that the three degrees of freedom for phi represents the three boosts and that the three degrees of freedom for theta represents the spatial rotations.

rmk. The submatrix of the matrix above is simply the rotation matrix, whose eigenvector for the eigenvalue of 0 is the principle axis constructable as follows.

$$R_{ij} = \begin{pmatrix} 0 & \theta_z & -\theta_y \\ -\theta_z & 0 & \theta_x \\ \theta_y & -\theta_x & 0 \end{pmatrix}_{ij} = \sum_{k=1}^{3} \epsilon_{ijk}\theta_k$$

Now, we find the eigenvector of the matrix above which shall satisfy

$$\sum_{j=0}^{3} R_{ij}\theta_j = 0$$

this vector would be exactly the principle axis, expressible as

$$\boldsymbol{\theta} = \theta \frac{\boldsymbol{\theta}}{\theta} = \theta \, \hat{\boldsymbol{\theta}}$$

4 Lecture 4

In this lecture, we consider the exponential map which parametrizes a transformation from one reference frame to another. Then, we define and investigate the transformation properties of various vectors in space-time, namely proper distance, proper time, and four velocity.

4.1 The Exponential Map

def. An exponential map is parametrized map from one reference frame to another. That is, $x'^{\mu} \to x'^{\mu}(x) = f^{\mu}(\lambda = 1, x)$, and $f^{\mu}(0, x) = x^{\mu}$. We now calculate the first

n-derivatives of this function.

$$\begin{cases} \frac{d}{d\lambda} f^\mu(\lambda, x) & = V^\mu(f(\lambda, x)) = V^\rho(f) \frac{\partial}{\partial f^\rho} f^\mu \\ \frac{d^2}{d\lambda^2} f^\mu(\lambda, x) & = \frac{d}{d\lambda} V^\mu(f(\lambda, x)) = \frac{\partial V^\mu}{\partial f^\nu} \frac{df^\nu}{d\lambda} \\ & = V^\nu \frac{\partial V^\mu(f)}{\partial f^\nu} \\ \frac{d^3}{d\lambda^3} f^\mu(\lambda, x) & = \frac{d}{d\lambda} \Big(V^\nu(f) \frac{\partial V^\mu(f)}{\partial f^\nu} \Big) \\ & = \frac{df^\rho}{d\lambda} \frac{\partial}{\partial f^\rho} \Big(V^\nu(f) \frac{\partial}{\partial f^\nu} V^\mu(f) \Big) \\ & = V^\rho(f) \frac{\partial}{\partial f^\rho} \Big(V^\nu(f) \frac{\partial}{\partial f^\nu} V^\nu(f) \Big) \\ & = \Big(V^\rho(f) \frac{\partial}{\partial f^\rho} \Big)^2 V^\mu(f) \\ \frac{d^n}{d\lambda^n} f^\mu(\lambda, x) & = \Big(V^\rho(f) \frac{\partial}{\partial f^\rho} \Big)^{n-1} V^\mu(f) \end{cases}$$

We can thus find the function above, using Taylor's expansion, to be

$$\begin{aligned} f^\mu(\lambda, x) &= \sum_{n=0}^\infty \frac{\lambda^n}{n!} \Big(\frac{d^n f^\mu(\lambda, x)}{d\lambda^n} |_{\lambda=0} \Big) \\ &= \sum_{n=0}^\infty \frac{\lambda^n}{n!} \Big(\Big(V^\rho(f) \frac{\partial}{\partial f^\rho} \Big)^{n-1} V^\nu(f) \Big) |_{\lambda=0} \\ &= \sum_{n=0}^\infty \frac{\lambda^n}{n!} \Big(\Big(V^\rho(x) \frac{\partial}{\partial x^\rho} \Big)^{n-1} V^\nu(x) \Big) \\ &= \sum_{n=0}^\infty \frac{\lambda^n}{n!} \Big(V^\rho(x) \frac{\partial}{\partial x^\rho} \Big)^n x^\mu = \exp\{\lambda V^\rho(x) \partial_\rho\} x^\mu \end{aligned}$$

which is approximately $x^\mu + \lambda V^\mu(x) + \cdots$. This tells us that $V^\mu(x) = \delta x^\mu$ and that the first derivative gives us information about infinitesimal difference as $x^\mu \to x'^\mu(x)$.

4.2 Four Vectors and their Transformations

def. We now define proper distance, which can be thought of as the invariant version of distance in space time.

$$ds^2 = -c^2dt^2 + dx \cdot dx$$

def. Proper time, in the other hand, is the above divided by c^2.

$$\begin{aligned} d\tau^2 &= dt^2 - dx \cdot dx/c^2 \\ &= dt'^2 - dx' \cdot dx'/c^2 \end{aligned}$$

Using this fact that there is this version of invariant time, we define covariant velocity.

def. Covariant velocity (four velocity)

$$V^\mu = \frac{dx^\mu}{d\tau}$$

It is worth noting how this covariant form of velocity transforms.

$$V'^\mu = \frac{dx'^\mu}{d\tau'} = L^\mu_\nu \frac{dx^\nu}{d\tau} = L^\mu_\nu V^\nu$$

where $L^\mu_\nu = dx'^\mu/dx^\nu$. Note that this quantity has the same transformation properties as the infinitesimal displacement vector, where

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu$$

This displacement vector transformed covariantly, thus considered a vector in space-time. As such, it can also be called a $(0, 1)$-tensor. Meanwhile, it is to be noted that partial derivatives transform oppositely (contravariantly),

$$\partial'_\mu = \frac{\partial x^\nu}{\partial x'^\nu} \partial_\nu$$

Such tensors are also called $(1, 0)$-tensors.

rmk. Noting that a vector can be raised or lowered an index using the space-time metric, we discover that lowered indices transform oppositely from when it is raised.

$$\begin{aligned} V'_\mu &= \eta_{\mu\nu} V'^\mu = \eta_{\mu\nu} L^\mu_\rho V^\rho \\ &= [(L^T)^{-1}\eta]_{\mu\rho} V^\rho = [(L^T)^{-1}]^\sigma_\mu \eta_{\sigma\rho} V^\rho \\ &= [(L^T)^{-1}]^\sigma_\mu V_\sigma \end{aligned}$$

In other notation,

$$
\begin{aligned}
\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\eta_{\rho\sigma} &= \eta_{\mu\nu} \\
V'_\mu = \eta_{\mu\nu}V'^\nu &= \eta_{\mu\nu}\frac{\partial x'^\nu}{\partial x^\rho}V^\rho \\
= \frac{\partial x^\rho}{\partial x'^\mu}\eta_{\rho\sigma}V^\rho &= \frac{\partial x^\rho}{\partial x'^\mu}V_\rho
\end{aligned}
$$

rmk. At this point, we remark that the inverse of tensors have lowered and raised indices, and that transposes simply change the order in which the indices take place.

5 Lecture 5

We now generalize the investigations of particular contravariant (sets of numbers that transform like the proper distance vector) and covariant (sets of numbers that transform inversely to the proper distance vector) vectors. After observing the transformation of a lot more vectors in space-time, we reconstruct the theory of electromagnetism in terms of tensors and motivate our reasoning for using tensors to describe physical phenomena: we want a framework invariant of reference frame (i.e., we want 0 to remain 0). We finish off by noting that gravitational potential only exist in rest or uniformly moving frames.

5.1 Contravariant and Covariant Vectors and their Transformations

recall. Consider the following coordinate transformation $x^\mu \to x'^\mu(x)$. We can consider the transformation for an infinitesimal

$$dx^\mu \to dx'^\mu = dx^\nu \frac{\partial x'^\mu}{\partial x^\nu}$$

through the direct application of the chain rule, we identify that the vector transforms covariantly. The inverse relation works for the partial, where

$$\partial_\mu \to \partial' \mu = \frac{\partial x^\nu}{\partial x'^\mu}\partial_v$$

we also consider the case for the Kronecker delta, and

$$\delta^\nu_\mu \to \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x'^\nu}{\partial x^\sigma}\delta^\sigma_\rho = \delta^\nu_\mu$$

lastly, for the case of the space-time metric,

$$\eta_{\mu\nu} \to \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\eta_{\rho\sigma} = \eta_{\mu\nu}$$

Consider how Poincare transformation works on lowered and raised indices.

$$\begin{aligned} dx_\mu = \eta_{\mu\nu} dx^\nu \to dx'_\mu &= \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\eta_{\rho\sigma}\frac{\partial x'^\nu}{\partial x^\lambda}dx^\lambda \\ &= \frac{\partial x^\rho}{\partial x'^\mu}\eta_{\rho\sigma}\delta^\sigma_\lambda dx^\lambda = \frac{\partial x^\rho}{\partial x'^\mu}dx_\rho \end{aligned}$$

for the partial,

$$\begin{aligned} \partial^\mu = \eta^{\mu\nu}\partial_\nu \to \partial'^\mu &= \frac{\partial x'_\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma}\eta^{\rho\sigma}\frac{\partial x^\lambda}{\partial x'^\nu}\partial_\lambda \\ &= \frac{\partial x'^\mu}{\partial x^\rho}\eta^{\rho\sigma}\delta^\lambda_\sigma\partial_\lambda = \frac{\partial x'^\mu}{\partial x^\rho}\eta^{\rho\lambda}\partial_\lambda = \frac{\partial x'^\mu}{\partial x^\rho}\partial^\rho \end{aligned}$$

for the proper time, however,

$$d\tau \to d\tau' = d\tau$$

thus motivating its use in the covariant form of velocity.

def. We define four velocity as the derivative of proper distance against proper time, and we can observe that it transforms contravariantly.

$$\frac{dx^\mu}{d\tau} \to \frac{dx'^\mu}{d\tau'} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau}$$

cor. A great property of the four-velocity is that it satisfies the following identity. Which comes directly from the definition of proper distance by dividing both sides by $d\tau$.

$$\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\eta_{\mu\nu} = -c^2$$

This fact is obvious once you consider x^μ and x^ν to move along the particle whose proper time is measured. The time derivatives of the spacial components would be zero and the result is trivially $-c^2$.

def. Four-momentum is simply four velocity times mass.

$$p_\mu = m\frac{dx_\mu}{d\tau}$$

cor. Contracting four momentum using the spacetime metric, we obtain the following familiar expression for energy.

$$
\begin{aligned}
p_\mu p_\nu \eta^{\mu\nu} &= p^2 = -m^2c^2 \\
(p_0)^2 - \boldsymbol{p}\cdot\boldsymbol{p} &= m^2c^2 \\
p_0 &= \sqrt{m^2c^2 + \boldsymbol{p}\cdot\boldsymbol{p}} \\
p_0 &= mc\sqrt{1 + \boldsymbol{p}\cdot\boldsymbol{p}/m^2c^2} \\
&\approx mc + \boldsymbol{p}\cdot\boldsymbol{p}/2mc + \cdots \\
E &\approx mc^2 + \boldsymbol{p}\cdot\boldsymbol{p}/2m + \cdots
\end{aligned}
$$

5.2 The Formulation of Electromagnetism using Tensors

rmk. The reason why we write the four-momentum with lowered indices is because it naturally arises in Lagrangian mechanics.

$$
p_\mu = \frac{\partial \mathcal{L}}{\partial_0 \partial x^\mu}
$$

def. The reason why we want to write physics in a covariant manner is because equations must be tensorial for things to be conserved. To display this, we investigate electromagnetism. Lorentz force in non-relativistic cases can be written as

$$
m\boldsymbol{a} = q\boldsymbol{v} \times \boldsymbol{B} + q\boldsymbol{E}
$$

In relativistic cases,

$$
m\frac{d^2x^\mu}{d\tau^2} = qF^\mu_\nu \frac{dx^\nu}{d\tau}
$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Note that it is empty in its diagonal ($F_{\mu\mu} = 0$) and is antisymmetric ($F_{\mu\nu} = -F_{\nu\mu}$). In expanded form,

$$
\begin{cases}
m\dfrac{d^2x^1}{d\tau^2} = qF^1_0\dfrac{dx^0}{d\tau} + qF^1_2\dfrac{dx^2}{d\tau} + qF^1_3\dfrac{dx^3}{d\tau} \\[2ex]
m\dfrac{d^2x^1}{d\tau^2} = qF^2_0\dfrac{dx^0}{d\tau} + qF^2_2\dfrac{dx^2}{d\tau} + qF^2_3\dfrac{dx^3}{d\tau} \\[2ex]
m\dfrac{d^2x^1}{d\tau^2} = qF^2_0\dfrac{dx^0}{d\tau} + qF^2_2\dfrac{dx^2}{d\tau} + qF^2_3\dfrac{dx^3}{d\tau} \\[2ex]
m\dfrac{d^2x^1}{d\tau^2} = qF^3_0\dfrac{dx^0}{d\tau} + qF^3_2\dfrac{dx^2}{d\tau} + qF^3_3\dfrac{dx^3}{d\tau}
\end{cases}
$$

where

$$F_{\mu\nu} = \begin{pmatrix} E_x/c & B_z & B_y & B_x \\ E_y/c & B_z & B_y & B_x \\ E_z/c & B_z & B_y & B_x \end{pmatrix}$$

prop. We derive that the electromagnetic potential transforms convariantly and that the tensor F transforms covariantly twice.

$$\begin{aligned} A_\mu(x) \to A'_\mu(x') &= \frac{\partial x^\lambda}{\partial x'^\mu} A_\lambda(x) \\ F_{\mu\nu}(x) \to F'_{\mu\nu}(x') &= \partial'_\mu A'_\nu - \partial'_\nu A'_\mu \\ &= \frac{\partial x^\rho}{\partial x'^\mu} \partial_\rho \Big(\frac{\partial x^\sigma}{\partial x'^\nu} A_\sigma \Big) - (\mu \leftrightarrow \nu) \\ &= \frac{\partial^2 x^\sigma}{\partial x'^\mu \partial x'^\nu} A_\sigma + \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \partial_\rho A_\sigma - (\mu \leftrightarrow \nu) \\ &= \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \partial_\rho A_\sigma - \frac{\partial x^\sigma}{\partial x'^\nu} \frac{\partial x^\rho}{\partial x'^\mu} \partial_\sigma A_\rho \\ &= \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} F_{\rho\sigma} \end{aligned}$$

HW. We can do something similar with the F tensor with one upper and one lower index, and it multiplied with four velocity.

$$\begin{aligned} F^\mu_\nu \to F'^\mu_\nu &= \eta^{\mu\kappa} F'_{\kappa\nu} \\ &= \frac{\partial x'^\mu}{\partial x^\alpha} \eta^{\alpha\beta} \frac{\partial x'^\kappa}{\partial x^\beta} \frac{\partial x^\rho}{\partial x'^\kappa} \frac{\partial x^\sigma}{\partial x'^\nu} F_{\rho\sigma} \\ &= \frac{\partial x'^\mu}{\partial x^\alpha} \eta^{\alpha\beta} \frac{\partial x^\sigma}{\partial x'^\nu} F^\alpha_\sigma \\ &= \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\sigma}{\partial x'^\nu} F^\alpha_\sigma \end{aligned}$$

HW. A simple corollary would be transformation for the F tensor times the four velocity

$$F^\mu_\nu \frac{dx^\nu}{d\tau} \to F'^\mu_\nu \frac{dx'^\nu}{d\tau} = \frac{\partial x'^\mu}{\partial \lambda} F^\lambda_\nu \frac{dx^\nu}{d\tau}$$

thm. Finally, we see how the Lorentz force transforms between reference frames.

$$\begin{aligned} m\frac{d^2 x^\mu}{d\tau^2} - qF^\mu_\nu \frac{dx^\nu}{d\tau} &= 0 \\ m\frac{d^2 x'^\mu}{d\tau} - qF'^\mu_\nu \frac{dx'^\nu}{d\tau} &= \frac{dx'^\mu}{dx^\lambda} \Big(m\frac{d^2 x^\lambda}{d\tau^2} - qF^\lambda_\nu \frac{dx^\nu}{d\tau} \Big) \\ &= 0 \end{aligned}$$

The highlight is that zero-force is remained that way in both reference frames.

cor. A simple corollary from above is that four-acceleration times four-velocity is zero. Observe that,

$$\frac{dx_\mu}{d\tau}\frac{d^2x^\nu}{d\tau^2} = \frac{dx_\mu}{d\tau}qF^\mu_\nu\frac{dx^\nu}{d\tau} = qF_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

as

$$\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\eta_{\mu\nu} = -c^2$$

cor. A following corollary is that in this framework, gravitational potential doesn't exist. We first define gravitational potential as

$$m\frac{d^2x^\mu}{d\tau^2} = -\partial^\mu V$$

However, suggesting that such potential exists would lead us to a non-vanishing four-acceleration times four-velocity.

$$\begin{aligned} m\frac{d^2x^\mu}{d\tau^2}\frac{dx^\nu}{d\tau}\eta_{\mu\nu} &= -\partial^\mu V\frac{dx_\mu}{d\tau} \\ &= -\frac{dx^\mu}{d\tau}\partial_\mu V \\ &= -\frac{d}{d\tau}V(x(\tau)) \end{aligned}$$

As the four velocity times four acceleration is always zero, either the derivative of the potential (which is acceleration) is zero or the derivative of the proper distance (which is velocity) is zero which implies that the reference frame we are referring to is either in rest or moving in constant speed.

6 Lecture 6

In this lecture, we rewrite Maxwell's equations in terms of tensors. After we do so, we prove Poincare's lemma for $(0,2)$*-tensors by stating an explicit form of the potential function, therefore proving that electromagnetic force has a potential function in all reference frames. We shortly deviate and prove Poincare's lemma for vector functions (which are* $(1,0)$*-tensors). We finish off by suggesting an alternate form of current density and thereof showing that the derivative current density is zero, ultimately proving that current density is a conserved throughout space-time.*

6.1 Maxwell's Equations in Terms of Tensors

def. Using the covariant framework that we have created, we can rewrite Maxwell's equations in terms of tensors as

$$\begin{cases} \partial_\mu F^{\mu\nu} = J^\nu \\ \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \end{cases}$$

Here, we note that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and that $J^\nu = (\rho/c, \boldsymbol{J})$.

6.2 Poincare's Lemma

thm. Poincare's lemma states that when a force is conservative, there is a potential function for that force.

pf. We can prove this by explicitly showing that such a function exits. We first claim that it has the following form.

$$A_\nu(x) = \int_0^1 sF_{\mu\nu}(x_s)\frac{dx_s^\mu}{ds}\,ds$$

Here, we use the notation $x_s = x(s)$. Note that the variable x is parametrized as a line segment in terms of s.

$$\begin{cases} x^\mu(s) = (x^\mu - x_0^\mu)s + x_0^\mu \\ x^\mu(s=1) = x^\mu \\ x^\mu(s=0) = x_0^\mu \end{cases}$$

To verify that this is indeed the potential function we ought to find, we derivate, obtaining

$$\begin{aligned} \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) &= \partial_\mu\Big[\int_0^1 sF_{\rho\nu}(x_s)\frac{dx_s^\rho}{ds}\,ds\Big] - (\mu \leftrightarrow \nu) \\ =&\int_0^1 s^2\frac{\partial}{\partial x_s^\mu}F_{\rho\nu}(x_s)\frac{dx^\rho}{ds} + sF_{\rho\nu}(x_s)\delta_\mu^\rho - s^2\frac{\partial}{\partial x_s^\nu}F_{\rho\mu}(x_s)\frac{dx^\rho}{ds} \\ &- sF_{\rho\mu}(x_s)\delta_\nu^\rho\,ds \\ =&\int_0^1 s^2\frac{\partial}{\partial x_s^\mu}F_{\rho\nu}\frac{dx^\rho}{ds} + sF_{\mu\rho}(x_s)\delta_\mu^\rho + s^2\frac{\partial}{\partial x_s^\nu}F_{\mu\rho}(x_s)\frac{dx^\rho}{ds} \\ &+ sF_{\mu\rho}(x_s)\delta_\nu^\rho\,ds \\ =&\int_0^1 s^2\frac{dx^\rho}{ds}\Big[\frac{\partial}{\partial x_s^\mu}F_{\rho\nu}(x_s) + \frac{\partial}{\partial x_s^\nu}F_{\mu\rho}(x_s)\Big] + 2sF_{\mu\nu}(x_s)\,ds \\ =&\int_0^1 -s^2\frac{\partial x^\rho}{\partial s}\frac{\partial}{\partial x_s^\rho}F_{\nu\mu} + 2sF_{\mu\nu}(x_s)\,ds \\ =&\int_0^1 \frac{\partial}{\partial s}\Big[s^2F_{\mu\nu}\Big]\,ds \\ =&F_{\mu\nu} \end{aligned}$$

rmk. In the constructed potential function above, we can add any function whose partial vanishes (such as $\partial_\mu\Lambda$ for any function Λ as $\partial_\mu\partial_\nu\Lambda - \partial_\nu\partial_\mu\Lambda = 0$). This is symmetry between fields are called Gauge symmetry.

pf. We can do something similar for vector functions where we claim that the potential has the following form.

$$V = \int_0^1 \frac{dx_s}{ds} f(x_s)\, ds$$

then,

$$\begin{aligned}
\partial_i V &= \partial_i \int_0^1 \frac{dx_s^j}{ds} f_j(x_s)\, ds \\
&= \int_0^1 f_i(x_s) + s\frac{dx_s^j}{ds}\frac{\partial f_i(x_s)}{\partial x_s^i}\, ds \\
&= \int_0^1 f_i(x_s) + s\frac{dx_s^j}{ds}\frac{\partial f_i(x_s)}{\partial x_s^j}\, ds \\
&= \int_0^1 f_i(x_s) + s\frac{df_i(x_s)}{ds}\, ds \\
&= \int_0^1 \frac{d}{ds}\Big[s f_i(x_s)\Big]\, ds \\
&= f_i(x)
\end{aligned}$$

6.3 Current Density as a Tensor

def. Consider a group of charged particles whose trajectories are parametrized by proper time, which we also will refer to as the worldline parameter (the trajectory is a function of proper time, $x_n^\mu(\tau)$). The current density vector can then be expressed as

$$J^\mu(x) = \sum_n \int q_n \frac{dx_n^\mu(\tau)}{d\tau} \delta^{(4)}(x - x_n(\tau))\, d\tau$$

here, $\delta^{(4)}(x - x(\tau)) = \delta(x^0 - x^0(\tau))\delta(x^1 - x^1(\tau)) \cdots \delta(x^3 - x^3(\tau))$. It is important to note that this expression is equivalent to the aforementioned expression $J^\mu = (\rho/c, \boldsymbol{J})$.

pf. We show that the derivative of the current density is 0, being a conservative quantity throughout space-time.

$$\begin{aligned}
\partial_\mu J^\mu &= \partial_\mu \Big[\sum_n \int q_n \frac{dx_n^\mu(\tau)}{d\tau} \delta^{(4)}(x - x_n(\tau))\, d\tau\Big] \\
&= \sum_n \int q_n \frac{dx_n^\mu(\tau)}{d\tau} \frac{\partial}{\partial x^\mu(\tau)} \delta^{(4)}(x - x_n(\tau))\, d\tau \\
&= \sum_n \int q_n \frac{d}{d\tau} \delta^{(4)}(x - x_n(\tau))\, d\tau \\
&= \sum_n q_n \delta^{(4)}(x - x_n(\tau)) \Big|_{\tau=-\infty}^{\tau=+\infty} \\
&= 0
\end{aligned}$$

def. We introduce the covariant derivative through observing what kind of derivations do not change the physics of wavefunctions despite phase change. Consider the transformation $\psi \to \psi e^{i\theta} = \psi'$. Schrodinger's equation becomes

$$i\hbar \frac{\partial}{\partial A}\psi = \frac{1}{2m}(i\hbar)^2\psi + V\psi$$

Schrodinger's equation only works when you define a new derivative (minimal coupling)

$$\partial_\mu \to D_\mu = \partial_\mu - iA_\mu$$

Here, we state that $A_\mu \to A'_\mu = A_\mu + \partial_\mu\theta$. Then,

$$\begin{aligned} D_\mu\psi \to D'_\mu\psi' &= \partial_\mu\psi' - iA'_i\psi' \\ &= e^{i\theta}D_\mu\psi \\ &= e^{i\theta}(\partial_\mu + i\partial_\mu\theta\psi - iA'_\mu\psi) \end{aligned}$$

7 Lecture 7

In this lecture, we define what tensors are (objects that transform contravariantly or covariantly) and introduce weights, which are how tensors scale when transforming. To finish off, we introduce the concept of Vierbeins, which are metrics that allow us to convert to locally falling frames.

recall. Before delving into the main topic, we again refer to how certain tensorial objects transform.–

$$\begin{aligned} V_\mu(x) \to V'_\mu(x') &= \frac{\partial x^\lambda}{\partial x'^\mu}V_\lambda(x) \\ V^\mu(x) = \eta^{\mu\nu}V_\nu(x) &\to V'^\mu(x') \\ \eta_{\mu\nu} &= \frac{\partial x^\rho}{\partial x'^\mu}\eta_{\rho\sigma}\frac{\partial x^\rho}{\partial x'^\nu} \\ \eta &= \left(\frac{\partial x}{\partial x'}\right)\eta\left(\frac{\partial x}{\partial x'}\right)^T \\ \eta^{-1} &= \left(\frac{\partial x'}{\partial x}\right)^T\eta^{-1}\left(\frac{\partial x'}{\partial x}\right) \\ \eta^{\mu\nu} &= \frac{\partial x'^\mu}{\partial x^\rho}\eta^{\rho\sigma}\frac{\partial x^\nu}{\partial x^\sigma} \end{aligned}$$

7.1 Tensors and their Weights

Using the above, we show how tensors with upper indices transform.

$$
\begin{aligned}
\eta^{\mu\nu}V'_\nu(x') &= \frac{\partial x'^\mu}{\partial x^\rho}\eta^{\rho\sigma}\frac{\partial x'^\nu}{\partial x^\sigma}\frac{\partial x^\lambda}{\partial x'^\nu}V_\lambda(x) \\
&= \frac{\partial x'^\mu}{\partial x^\rho}\eta^{\rho\sigma}V_\sigma(x) \\
V'^\mu(x') &= \frac{\partial x'^\mu}{\partial x^\rho}V^\rho(x) \\
&= \Lambda^\mu_\rho V^\rho
\end{aligned}
$$

cor. We can do the same process as above and show how lower indices transform using how upper indices transform.

$$
\begin{aligned}
V'^\mu &= \Lambda^\mu_\rho V^\rho \\
V'_\mu &= (\Lambda^{-1})^\lambda_\mu V_\lambda = V_\lambda(\Lambda^{-1})^\lambda_\mu \\
&= \Lambda^\lambda_\mu V_\lambda
\end{aligned}
$$

Note that

$$
\begin{aligned}
\Lambda^T\eta\Lambda &= \eta \\
\eta\Lambda &= (\Lambda^T)^{-1}\eta \\
\eta\Lambda\eta^{-1} &= (\Lambda^{-1})^T \\
\Lambda^\rho_\mu &= (\Lambda^{-1})^\mu_\rho
\end{aligned}
$$

rmk. In representation theory, we look at matrix representations of group elements (in this case coordinate transformations). In reference to the matrices above, we can see how the matrix representation of a certain transformation is equal to another transformation's (the inverse transformation) transpose. Mathematically, $g \implies M(g) = (M^{-1}(g))^T$. In this way, the group of inverse matrices' transpose model the original group's behavior.

$$
\begin{aligned}
g_1g_2 &= g_3 \\
M(g_1)M(g_2) &= M(g_3) \\
M(g_2)^{-1}M(g_1)^{-1} &= M(g_3)^{-1} \\
(M(g_1)^{-1})^T(M(g_2)^{-1})^T &= (M(g_3)^{-1})^T
\end{aligned}
$$

def. We define a (p,q) tensor to be a list of components that have p upper indices and q lower indices.

$$
T^{\mu_1\mu_2\cdots\mu_p}_{\nu_1\nu_2\cdots\nu_q}(x) \to T'^{\mu_1\mu_2\cdots\mu_p}_{\nu_1\nu_2\cdots\nu_q}(x')
$$

which is equal to

$$||\frac{\partial x'}{\partial x}||^w \frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}} \frac{\partial x'^{\mu_2}}{\partial x^{\rho_2}} \cdots \frac{\partial x'^{\mu_p}}{\partial x^{\rho_p}} T^{\rho_1 \cdots \rho_p}_{\sigma_1 \cdots \sigma_q} \frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\sigma_q}}{\partial x'^{\nu_q}}$$

def. We define the weight w as

$$||\frac{\partial x'}{\partial x}||^w = ||\frac{\partial x}{\partial x'}||^{-w}$$

recall. We defined the current density as

$$\begin{aligned} J_\mu(x) &= \sum_n \int_{-\infty}^{\infty} q_n \frac{\partial x_n^\mu(\tau)}{\partial \tau} \delta^{(4)}(x - x_n(\tau))\, d\tau \\ &= \sum_n q_n \int_{-\infty}^{\infty} dx_n^0 \frac{dx^\mu}{dx_n^0} \delta^{(3)}(x - x_n)\, dx_n^0 \\ &= \sum_n q_n \frac{dx_n^\mu}{dx_n^0} \delta^{(3)}(x - x_n) \end{aligned}$$

this is equal to

$$= \begin{cases} \sum_n q_n \delta^{(3)}(x - x_n) = \rho \\ \\ \sum_n q_n \frac{dx_n}{dt} \delta^{(3)}(x - x_n) = \boldsymbol{J} \end{cases}$$

This is due to the fact that

$$\int \delta(x-y) f(x)\, dx = f(y)$$

$$\int \frac{\partial x}{\partial x'} \delta(x-y) f(x(x'))\, dx' = f(y(y'))$$

implying

$$\delta^{(4)}(x' - y') = ||\frac{\partial x}{\partial x'}||^1 \delta^{(4)}(x-y)$$

When the weight is 1, we refer a tensor to be a scalar density while referring to tensors with higher weights as vector densities. Note that we are generalizing the notion of tensors here. We are also including objects that scale as they transform.

recall. We know for a fact that $x^\mu \to x'^\mu(x)$,

$$dx^\mu \to dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu$$

In general relativity, we state that the metric is g in general cases, a function of x which is symmetric under the lower indices. It is a $(0, 2)$ tensor.

$$g_{\mu\nu}(x) = g_{\nu\mu}(x)$$

After the coordinate transformation, we state that

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\rho}{\partial x'^\nu} g_{\rho\sigma}(x)$$

and

$$g^{\mu\nu}(x) \to g'^{\mu\nu}(x') = \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} g^{\rho\sigma}(x)$$

Note that

$$g = ||g_{\mu\nu}|| \to g' = ||\frac{\partial x}{\partial x'}||^2 g$$

and

$$\sqrt{-g} \to \sqrt{-g'} = ||\frac{\partial x}{\partial x'}||\sqrt{-g}$$

which tells us that the negative square root is a scalar density with $w = 1$. The following is thus a tensor with a weight of 0.

$$\delta^{(4)}(x)/\sqrt{-g}$$

7.2 Verbeins

def. We now introduce the concept of Verbeins. Simply put, Verbeins are transformations from a curved frame to a local Lorentzian (flat) frame which we shall denote as y^μ. We say local from the fact that this definition only suffices locally as tidal forces occur as you move further from the exact point that we are transforming from. As you move slightly from one direction to another, you are no longer in a inertial state. This coordinate system we are transforming to is also called the Riemann normal coordinate system. The proper distance in this frame would be given as

$$\begin{aligned} ds^2 = \eta_{\mu\nu} dy^\mu dy^\nu &= \eta_{\mu\nu} dx^\rho \frac{\partial y^\mu}{\partial x^\rho} dx^\sigma \frac{\partial y^\nu}{\partial x^\sigma} \\ &= dx^\rho dx^\sigma \eta_{\mu\nu} \frac{\partial y^\mu}{\partial x^\rho} \frac{\partial y^\nu}{\partial x^\sigma} \\ &= dx^\rho dx^\sigma g_{\rho\sigma} \end{aligned}$$

Note that the function g would be a function of x^μ as the transformation to the locally inertial frame would vary from point to point.

def. We mathematically formally define verbiens as

$$g_{\rho\sigma}(x) = e^a_\rho(x) e^b_\sigma(x) \eta_{ab}$$

a, b runs through 0,1,2, and 4. We can also express veribeins as

$$e^a_\mu(X) = \frac{\partial y^a}{\partial x^\mu}|_{x=X}$$

In this perspective, verbeins are functions that take in points in space-time and give transformations from curved spacetime coordinates to locally flat coordinates.

8 Lecture 8

In this lecture, we use the verbeins to define the Christoffel symbol, which show how basis objects transform.

8.1 More on Verbeins

def. Using the space-time metric, proper distance is given by

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$$

In turn, the metric in terms of Vierbeins can be written as

$$g_{\mu\nu}(x) = e^a_\mu(x)e^b_\nu(x)\eta_{ab}$$

Here, vierbeins are functions of x^μ as

$$e^a_\mu(X) = \frac{\partial y^a_X(x)}{\partial x^\mu}\Big|_{x=X}$$

cor. A simple corollary to our new definition of proper distance is that proper time is given as

$$d\tau^2 = -\frac{1}{c^2}ds^2 = -\frac{1}{c^2}dx^\mu dx^\nu g_{\mu\nu}$$

rmk. A special fact about verbeins are that they transform like tensors for the alphabetical indices as we can easily change the Lorentz coordinate system we are transforming from. Thus,

$$e^a_\mu(x) \to e'^a_\mu = e^b_\mu L^a_b(x)$$

The space-time metric, however, would remain invariant, $e^a_\mu e^b_\nu \eta_{ab} = e'^a_\mu e'^b_\nu \eta_{ab}$, as

$$L^c_a L^d_b \eta_{cd} = \eta_{ab}$$

def. In this manner, we can reformulate all transformations that we have mentioned in the following manner.

$$\begin{aligned}
x^\mu &\to x'^\mu(x) \\
dx^\mu &\to dx'^\mu = \frac{\partial x'^\mu}{\partial x^\rho}dx^\rho \\
d\tau &\to d\tau' = g_{\mu\nu}(x)dx^\mu dx^\nu = g'_{\mu\nu}(x')dx'^\mu dx'^\nu \\
g_{\mu\nu}(x) &\to g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\rho\sigma}(x) \\
\frac{dx^\mu}{d\tau} &\to \frac{dx'^\mu}{d\tau} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau}
\end{aligned}$$

8.2 The Christoffel Symbol

prop. We now consider how acceleration transforms from one frame to another and find the necessity of another geometrical object.

$$\begin{aligned}
\frac{d^2x^\mu}{d\tau^2} \to \frac{d^2x'^\mu}{d\tau^2} &= \frac{d}{d\tau}\Big(\frac{\partial x'^\mu}{\partial x^\lambda}\frac{dx^\lambda}{d\tau}\Big) \\
&= \frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial^2 x^\lambda}{\partial \tau^2} + \frac{d}{d\tau}\Big(\frac{\partial x'^\mu}{\partial x^\rho}\Big)\frac{dx^\rho}{d\tau} \\
&= \frac{\partial x'^\mu}{\partial x^\lambda}\frac{d^2 x^\lambda}{d^2\tau} + \frac{dx^\sigma}{d\tau}\frac{dx^\rho}{d\tau}\frac{\partial^2 x'^\mu}{\partial x^\sigma \partial x^\rho} \\
&= \frac{\partial x'^\mu}{\partial x^\lambda}\ddot{x}^\lambda + \dot{x}^\sigma \dot{x}^\rho \frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\sigma} \\
&= \frac{\partial x'^\mu}{\partial x^\lambda}\Big(\ddot{x}^\lambda + \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\sigma}\dot{x}^\rho \dot{x}^\sigma\Big)
\end{aligned}$$

where we use the notation $\dot{x}^\mu = dx^\mu/d\tau$. We can alternatively express the above using the coordinate system of a locally lorentz frame as the frame we are transforming to, and acceleration would be

$$\frac{d^2y^a}{d\tau^2} = \frac{\partial y^a}{\partial x^\lambda}\Big(\ddot{x}^\lambda + \frac{\partial x^\lambda}{\partial y^c}\frac{\partial^2 y^c}{\partial x^\rho \partial x^\sigma}\dot{x}^\rho \dot{x}^\sigma\Big)$$

The term we are interested in, and want to turn into in terms of the metric is,

$$\begin{aligned}
\frac{\partial x^\lambda}{\partial y^c}\frac{\partial^2 y^c}{\partial x^\rho \partial x^\sigma} &= e^\lambda_c \frac{\partial}{\partial x^\rho} e^c_\sigma \\
&= e^\lambda_b \delta^b_c \frac{\partial}{\partial x^\rho} e^c_\sigma \\
&= e^\lambda_b e_{c\kappa} e^{\kappa b} \frac{\partial}{\partial x^\rho} e^c_\sigma \\
&= g^{\lambda\kappa} e_{\kappa c}\Big(\frac{\partial}{\partial x^\rho} e^c_\sigma\Big) \\
&= -g^{\lambda\kappa}(\partial_\rho e_{\kappa c}) e^c_\sigma + g^{\lambda k}\frac{\partial}{\partial x^\rho} g_{\kappa\sigma} \\
&= -g^{\lambda\kappa}(\partial_k e_{\rho c}) e^c_\sigma + g^{\lambda\kappa}\frac{\partial}{\partial x^\rho} g_{\kappa\sigma}
\end{aligned}$$

There are a few points refer to. In moving from the second line to the third, we expanded the Kronecker delta into two verbien terms which follow from the fact that

$$e_{\kappa c} e^{b\kappa} = e^a_\kappa \eta_{ac} \eta^{bd} e^\kappa_d$$

with a being a free variable,

$$e_{\kappa c} e^{b\kappa} = e^d_\kappa \eta_{dc} \eta^{bd} e^\kappa_d = \delta^b_c$$

Therefore,

$$\begin{aligned}
\frac{d^2y^a}{d\tau^2} &= \frac{\partial y^a}{\partial x^\lambda}\Big[\ddot{x}^\lambda + g^{\lambda\kappa}\Big(\partial_\rho g_{\kappa\sigma} - (\partial_\kappa e_{\rho c})e^c_\sigma\Big)\dot{x}^\rho\dot{x}^\sigma\Big] \\
&= \frac{\partial y^a}{\partial x^\lambda}\Big[\ddot{x}^\lambda + g^{\lambda\kappa}\Big(\partial_\rho g_{\kappa\sigma} - (\partial_\kappa e_{\rho c}e^c_\sigma) + e_{\rho c}(\partial_\kappa e^c_\sigma)\Big)\Big] \\
&= \frac{\partial y^\lambda}{\partial x^\lambda}\Big[\ddot{x}^\lambda + g^{\lambda\kappa}\Big(\partial_\rho g_{\kappa\sigma} - \partial_\kappa g_{\rho\sigma} + (\partial_\kappa e_{\rho c}e^c_\sigma) - e^c_\sigma(\partial_\kappa e_{\rho c})\Big)\Big] \\
&= \frac{\partial y^a}{\partial x^\lambda}\Big[\ddot{x}^\lambda + \frac{1}{2}g^{\lambda\kappa}\Big(\partial_\rho g_{\kappa\sigma} - \partial_\kappa g_{\rho\sigma} + \partial_\sigma g_{\kappa\rho}\Big)\dot{x}^\rho\dot{x}^\sigma\Big] \\
&= \frac{\partial y^a}{\partial x^\lambda}\Big[\ddot{x}^\lambda + \frac{1}{2}g^{\lambda\kappa}(\partial_\rho g_{\kappa\sigma} + \partial_\sigma g_{\kappa\rho} - \partial_\kappa g_{\rho\sigma})\dot{x}^\rho\dot{x}^\sigma\Big] \\
&= \frac{\partial y^a}{\partial x^\lambda}\Big(\ddot{x}^\lambda + \Gamma^\lambda_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma\Big)
\end{aligned}$$

def. The symbol gamma is referred as the Christoffel symbol, the diffeomorhpism connection, or the affine connection.

$$\Gamma^\lambda_{\mu\nu}(x) = \frac{1}{2}g^{\lambda\rho}\Big(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\Big)$$

lem. Note that for the second line to the third line, we have utilized the following lemma for tensors symmetric in the lower two indices.

$$\begin{aligned}
A_{\mu\nu}\dot{x}^\mu\dot{x}^\nu &= A_{\nu\mu}\dot{x}^\nu\dot{x}^\mu \\
&= A_{\nu\mu}\dot{x}^\mu\dot{x}^\nu \\
&= \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})\dot{x}^\mu\dot{x}^\nu
\end{aligned}$$

def. Note that in free fall, the acceleration term would become zero.

$$\frac{d^2y}{d\tau^2} = 0$$

We thus obtain the following equation

$$\ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$$

rmk. In most the space-time metric can be approximated by the minkowskian metric and an additional pertubation term.

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

thm. We now show that we can approximate Newton's gravitational potential through assuming low speeds. In the majority of times, we refer to a stationary reference frame with $\dot{x}^\mu = (\dot{x}, \dot{\boldsymbol{x}}) \approx (c, 0, 0, 0)$. Thus, we are capable to approximate the geodesic equation as

$$0 = \ddot{x}^i + \Gamma^i_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \approx \ddot{x}^i + c^2\Gamma^i_{00}$$

where

$$\Gamma^i_{00} = \frac{1}{2}g^{i\lambda}\Big(\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00}\Big)$$

which is approximately

$$\approx -\frac{1}{2}\partial^i g_{00}$$

substituting, we can approximate a particle's acceleration as

$$\ddot{x}^i \approx -c^2\Gamma^i_{00} \approx \frac{1}{2}c^2\partial^i g_{00}$$

we hereof state without explanation that when solving this equation, at low speeds, we have the metric to be approximately

$$g_{00} \approx -1 - \frac{2\Phi}{c^2}$$

which is Newton's gravitational potential.

9 Lecture 9

9.1 Calculus of Variations

recall. We said that the acceleration of the LLF can be expressed as

$$\ddot{y}^a = e^a_\mu\Big[\ddot{x}^\mu + \Gamma^\mu_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma\Big]$$

Where the Christoffel symbol can be written as

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2}g^{\mu\nu}(\partial_\rho g_{\nu\sigma} + \partial_\sigma g_{\rho\nu} - \partial_\nu g_{\rho\sigma})$$

Also recall that the space-time metric can be written as

$$g_{\mu\nu}(x) = e^a_\mu(x)e^b_\nu(x)\eta_{ab}$$

where

$$e^a_\mu(X) = \frac{\partial y^a}{\partial x^\mu}\Big|_{x=X}$$

prop. Consider a line parametrized by a parameter lambda. The end points are denoted as

λ_1 and λ_2. The path of shortest length can be described by the following equation.

$$
\begin{aligned}
0 &= \delta \int_{\lambda_1}^{\lambda_2} \sqrt{\dot{x}^2 + \dot{y}^2}\, d\lambda \\
&= \int_{\lambda_1}^{\lambda_2} \frac{2\dot{x}\delta\dot{x} + 2\dot{y}\delta\dot{y}}{2\sqrt{\dot{x}^2 + \dot{y}^2}}\, d\lambda \\
&= -\int_{\lambda_1}^{\lambda_2} \delta x \frac{d}{d\lambda}\Big(\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\Big) + \delta y \frac{d}{d\lambda}\Big(\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\Big)\, d\lambda
\end{aligned}
$$

the last term follows from integration by parts. We now know that the terms in the parentheses vanish (we further assumed that $\delta x = 0$ at $\lambda = \lambda_1$ or $\lambda = \lambda_2$).

rmk. We want to show that the parameter lambda is best set as distance. Consider taking $\lambda \to \lambda'(\lambda)$, and

$$
\begin{cases}
d\lambda &= \dfrac{d\lambda}{d\lambda'} d\lambda' \\[2ex]
\dfrac{dx}{d\lambda} &= \dfrac{d\lambda'}{d\lambda}\dfrac{dx}{d\lambda'}
\end{cases}
$$

$$
\begin{aligned}
\int_{\lambda_1}^{\lambda_2} \sqrt{\Big(\frac{dx}{d\lambda}\Big)^2 + \Big(\frac{dy}{d\lambda}\Big)^2}\, d\lambda &= \int_{\lambda_1'}^{\lambda_2'} \sqrt{\Big(\frac{dx}{d\lambda'}\Big)^2 + \Big(\frac{dy}{d\lambda'}\Big)^2}\, d\lambda \\
&= \sqrt{dx^2 + dy^2}\Big|_1^2
\end{aligned}
$$

Thus we choose the parameter λ to be distance for the final line. With a Gauge choice of $\sqrt{\dot{x}^2 + \dot{y}^2} = 1$.

prop. We now finally use the variational principle to derive the geodesic equation.

$$
\begin{aligned}
S &= \int \sqrt{-g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu}\, d\lambda \\
\delta S &= \int \frac{\delta(-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)}{2\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}\, d\lambda \\
&= \int \frac{-\delta x^\lambda \partial_\lambda g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - g_{\mu\nu}\delta\dot{x}^\mu\dot{x}^\nu - g_{\mu\nu}\dot{x}^\mu\delta\dot{x}^\nu}{2\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}\, d\lambda \\
&= \int \frac{-\delta x^\lambda \partial_\lambda g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}{2\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}} + \delta x^\mu \frac{d}{d\lambda}\Big(\frac{g_{\mu\nu}\dot{x}^\nu}{2\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}\Big) + \delta x^\nu \frac{d}{d\lambda}\Big(\frac{g_{\mu\nu}\dot{x}^\mu}{2\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}\Big)\, d\lambda \\
&= \int -\frac{1}{2}\delta x^\mu \partial_\lambda g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{1}{2}\delta x^\mu \frac{d}{d\lambda}\Big(g_{\mu\nu}\dot{x}^\nu\Big) + \frac{1}{2}\delta x^\nu \frac{d}{d\lambda}\Big(g_{\mu\nu}\dot{x}^\mu\Big)\, d\lambda \\
&= \int -\frac{1}{2}\delta x^\mu \partial_\lambda g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \delta x^\mu(g_{\mu\nu}\ddot{x}^\nu + \dot{x}^\lambda\partial_\lambda g_{\mu\nu}\dot{x}^\nu)\, d\lambda \\
&= \int \delta x^\mu g_{\mu\nu}\Big(\ddot{x}^\nu + \frac{1}{2}g^{\nu\rho}(\partial_\lambda g_{\rho\sigma} + \partial_\sigma g_{\rho\lambda} - \partial_\rho g_{\lambda\sigma})\dot{x}^\nu)\Big)\, d\lambda
\end{aligned}
$$

10 Lecture 10

In this lecture we create tensorial version of a derivative (the covariant derivative).

10.1 The Covariant Derivative

recall.

$$
\begin{aligned}
V^\mu(x) \to V'^\mu(x') &= \frac{\partial x'^\mu}{\partial x^\rho} V^\rho(x) \\
\partial_\lambda V^\mu \to \partial'_\lambda V'^\mu &= \frac{\partial x^\sigma}{\partial x'^\lambda} \frac{\partial}{\partial x^\sigma} \Big(\frac{\partial x'^\mu}{\partial x^\rho} V^\rho \Big) \\
&= \frac{\partial x^\sigma}{\partial x'^\lambda} \frac{\partial x'^\mu}{\partial x^\rho} \partial_\sigma V^\rho + \frac{\partial x^\sigma}{\partial x'^\lambda} \frac{\partial^2 x'^\mu}{\partial x^\sigma \partial x^\rho} V^\rho \\
\partial_\lambda V_\mu \to \partial'_\lambda V'_\mu &= \frac{\partial x^\sigma}{\partial x'^\lambda} \partial_\sigma \Big(\frac{\partial x^\rho}{\partial x'^\mu} V_\rho \Big) \\
&= \frac{\partial x^\sigma}{\partial x'^\lambda} \frac{\partial x^\rho}{\partial x'^\mu} \partial_\sigma V_\rho + \frac{\partial^2 x^\rho}{\partial x'^\lambda \partial x'^\mu} V_\rho
\end{aligned}
$$

lem. In this lecture we prove the following lemma for second derivatives.

$$
\begin{aligned}
\frac{\partial^2 x^\rho}{\partial x'^\lambda x'^\mu} &= \frac{\partial x^\nu}{\partial x'^\lambda} \frac{\partial}{\partial x^\nu} \Big(\frac{\partial x^\rho}{\partial x'^\mu} \Big) \\
&= \frac{\partial x^\nu}{\partial x'^\lambda} \frac{\partial x^\rho}{\partial x'^\alpha} \frac{\partial^2 x'^\alpha}{\partial x^\nu \partial x^\beta} \frac{\partial x^\beta}{\partial x'^\mu}
\end{aligned}
$$

and thus

$$
\frac{\partial x'^\nu}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x'^\lambda \partial x'^\mu} = \frac{\partial x^\alpha}{\partial x'^\lambda} \frac{\partial x^\beta}{\partial x'^\mu} \frac{\partial^2 x'^\nu}{\partial x^\alpha \partial x^\beta}
$$

not different from

$$
\partial M^{-1} = M^{-1} \partial M M^{-1}
$$

and

$$
\partial (M M^{-1}) = 0
$$

prop. In this lecture we will consider $\partial_\lambda g_{\mu\nu} \to \partial'_\lambda g'_{\mu\nu}$ and see how the metric transforms.

$$
\begin{aligned}
\partial'_\lambda g'_{\mu\nu} &= \frac{\partial x^\gamma}{\partial x'^\lambda} \frac{\partial}{\partial x^\nu} \Big(\frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \Big) \\
&= \frac{\partial x^\gamma}{\partial x'^\lambda} \frac{\partial x^\alpha}{\partial x'^\beta} \frac{\partial x^\mu}{\partial x'^\nu} \partial_\gamma g_{\alpha\beta} + \Big(\frac{\partial^2 x^\alpha}{\partial x'^\lambda \partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} + \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial^2 x^\beta}{\partial x'^\lambda \partial x'^\gamma} \Big) g_{\alpha\beta}
\end{aligned}
$$

now, calculating the transformed Christoffel symbols,

$$
\begin{aligned}
\Gamma^{\lambda}_{\mu\nu} =& \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\rho\gamma} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \\
\Gamma'^{\lambda}_{\mu\nu} =& \frac{1}{2} g'^{\lambda\rho}(\partial'_\mu g'_{\rho\nu} + \partial'_\nu g'_{\mu\rho} - \partial'_\rho g'_{\mu\nu}) \\
=& \frac{1}{2} \frac{\partial x'^\lambda}{\partial x^\alpha} \frac{\partial x'^\rho}{\partial x^\beta} g^{\alpha\beta} \Big[\frac{\partial x^\gamma}{\partial x'^\mu} \frac{\partial x^k}{\partial x'^\rho} \frac{\partial x^\sigma}{\partial x'^\nu} \partial_\gamma g_{k\sigma} (\partial_\gamma g_{k\sigma} + \partial_\sigma g_{k\gamma} - \partial_k g_{\gamma\sigma}) \\
& + \Big(\frac{\partial^2 x^\alpha}{\partial^2 x'^\mu \partial x'^\nu} \frac{\partial x^\beta}{\partial x'^\nu} + \frac{\partial x^\alpha}{\partial x'^\rho} \frac{\partial^2 x^\beta}{\partial x;^\mu \, \partial x'^\nu} + \frac{\partial^2 x^\alpha}{\partial x'^\nu \partial x'^\rho} \frac{\partial x^\beta}{\partial x'^\mu} + \frac{\partial x^\alpha}{\partial x'^\rho} \frac{\partial^2 x^\beta}{\partial x'^\nu \partial x'^\mu} \\
& - \frac{\partial x^2 x^\alpha}{\partial x'^\rho \partial x'^\nu} \frac{\partial x^\beta}{\partial x'^\mu} - \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial^2 x^\beta}{\partial x'^\rho \partial x'^\mu} \Big) g_{\alpha\beta} \Big] \\
=& \frac{1}{2} \frac{\partial x'^\lambda}{\partial x^\alpha} \frac{\partial x^\gamma}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g^{\alpha k} (\partial_\gamma g_{k\sigma} + \partial_\sigma g_{k\gamma} - \partial_k g_{\gamma\sigma}) + \frac{\partial x'^\lambda}{\partial x^\phi} \frac{\partial x'^\rho}{\partial x^\zeta} g^{\phi\zeta} \frac{\partial x^\alpha}{\partial x'^\rho} \frac{\partial^2 x^\beta}{\partial x'^\mu \partial x'^\mu \partial x'^\nu} g_{\alpha\beta} \\
=& \frac{\partial x'^\lambda}{\partial x^\alpha} \frac{\partial x^\gamma}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \Gamma^\alpha_{\gamma\sigma} + \frac{\partial x'^\lambda}{\partial x^\beta} \frac{\partial^2 x^\beta}{\partial x'^\mu \partial x'^\nu} \\
=& \frac{\partial x'^\lambda}{\partial x^\alpha} \Big(\frac{\partial x^\beta}{\partial x'^\mu} \frac{\partial x^\gamma}{\partial x'^\nu} \Gamma^\alpha_{\beta\gamma} + \frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu} \Big)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\partial_\mu V_\nu \to \partial'_\mu V'_\nu =& \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \partial_\alpha V_\beta + \frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu} V_\alpha \\
=& \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \partial_\alpha V_\beta + \Big(\frac{\partial x^\alpha}{\partial x'^\lambda} \Gamma'^\lambda_{\mu\nu} - \frac{\partial x^\beta}{\partial x'^\mu} \frac{\partial x^\gamma}{\partial x'^\nu} \Gamma^\alpha_{\beta\gamma} \Big) V_\alpha \\
=& \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \Big(\partial_\alpha V_\beta - \Gamma^\gamma_{\alpha\beta} V_\gamma \Big) + \Gamma'^\lambda_{\mu\nu} V'_\lambda
\end{aligned}
$$

We thus identify a covariant form of the partial derivatives

$$
\partial'_\mu V'_\nu - \Gamma'^\rho_{\mu\nu} V'_\rho = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \Big(\partial_\alpha V_\beta - \Gamma^\gamma_{\alpha\beta} V_\gamma \Big)
$$

Thus we define the covariant derivative to be

$$
\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\lambda_{\mu\nu} V_\lambda
$$

With the vector having a upper index,

$$
\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda
$$

thus for an arbitrary tensor T,

$$
\nabla_\lambda T^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_q} = \partial_\lambda T^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_q} - w \Gamma^\rho_{\lambda\rho} T^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_q} T^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_q} + \sum_{i=1}^{P} \Gamma^{\mu_i}_{\lambda\rho} T^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_q} - \sum_{j=1}^{q} \Gamma^\rho_{\lambda\nu_j} T^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_q}
$$

11 Lecture 11

In this lecture, we learn a central assumption regarding general relativity, which is that there is no torsion in spacetime. This is equivalent to saying that the Christoffel symbol is symmetric on the lower indices. We also learn how the metric is covariantly constant, meaning that the covariant derivative of the metric is zero. We finish by showing that there always exists a frame such that the Christoffel symbol is zero. This frame is also called the Riemann normal coordinate system.

recall. Last class, we saw how the Christoffel symbol transformed, namely

$$\Gamma^{\lambda}_{\mu\nu} \to \Gamma'^{\lambda}_{\mu\nu}(x') = \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial x'^{\lambda}}{\partial x^{\kappa}} \Gamma^{\kappa}_{\rho\sigma}(x) + \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}} \frac{\partial x'^{\lambda}}{\partial x^{\rho}}$$

the partial, on the other hand,

$$\begin{aligned} \partial_{\mu} V^{\nu}(x) \to \partial'_{\mu} V'^{\nu}(x') &= \frac{\partial x^{\rho}}{\partial x'^{\rho}} \Big(\frac{\partial x'^{\nu}}{\partial x^{\sigma}} V^{\sigma} \Big) \\ &= \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x'^{\nu}}{\partial x^{\sigma}} \partial_{\rho} V^{\sigma} + \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial^2 x'^{\nu}}{\partial x^{\rho} \partial x^{\sigma}} V^{\sigma} \end{aligned}$$

The second term of the last equation is equal to

$$\begin{aligned} \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\rho}} \Big(\frac{\partial x'^{\nu}}{\partial x^{\sigma}} \Big) &= \frac{\partial}{\partial x'^{\mu}} \Big(\frac{\partial x'^{\nu}}{\partial x^{\sigma}} \Big) \\ &= - \frac{\partial x'^{\kappa}}{\partial x^{\alpha}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\kappa}} \frac{\partial x'^{\nu}}{\partial x^{\rho}} \end{aligned}$$

and thus we obtain, continuing from the partial transformation,

$$= \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x'^{\nu}}{\partial x^{\sigma}} \partial_{\rho} V^{\sigma} - \frac{\partial x'^{\kappa}}{\partial x^{\sigma}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\kappa}} \frac{\partial x'^{\nu}}{\partial x^{\rho}} V^{\sigma}$$

11.1 Torsionlessness

def. In general relativity we assume no torsion, with $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$. We define torsion to be $\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}$ (thus the Christofell symbol is symmetric in its lower indices).

11.2 Metric Compatibility

rmk. We state as a fact that $\nabla_{\lambda} g_{\mu\nu} = 0$ and that the metric is covariantly constant ($\iff$ ∇_{λ} and $g_{\mu\nu}$ are compatible). In this case, we say that the "connection" is torsionless. Then, the "connection" is the Christoffel symbol. Consider adding

$$\nabla_{\lambda} g_{\mu\nu} = \partial_{\lambda} g_{\mu\nu} - \Gamma^{\rho}_{\lambda\mu} g_{\rho\nu} - \Gamma^{\rho}_{\lambda\nu} g_{\mu\rho}$$

and

$$+\nabla_{\mu} g_{\nu\lambda} = \partial_{\mu} g_{\nu\lambda} - \Gamma^{\rho}_{\mu\nu} g_{\rho\lambda} - \Gamma^{\rho}_{\mu\lambda} g_{\nu\rho}$$

and

$$-\nabla_\nu g_{\lambda\mu} = \partial_\nu g_{\lambda\mu} - \Gamma^\rho_{\nu\lambda} g_{\rho\mu} - \Gamma^\rho_{\nu\mu} g_{\lambda\rho}$$

which is equal to

$$0 = \nabla_\lambda g_{\mu\nu} + \nabla_\mu g_{\nu\lambda} - \nabla_\nu g_{\lambda\mu} = \partial_\lambda g_{\mu\nu} + \partial_\mu g_{\nu\lambda} - \partial_\nu g_{\lambda\mu} - 2\Gamma^\rho_{\lambda\mu} g_{\rho\nu}$$

and we thus conclude

$$\Gamma^\nu_{\lambda\mu} = \frac{1}{2} g^{\nu\rho} \Big(\partial_\lambda g_{\rho\mu} + \partial_\mu g_{\lambda\rho} - \partial_\rho g_{\lambda\mu} \Big)$$

HW. The homework was to vertify that $\nabla_\lambda g_{\mu\nu} = 0$ by inserting the definition of Gamma into the equation above. Plugging our definition of the Christoffel symbol into the first equation we obtain

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \frac{1}{2} g^{\rho\nu} (\partial_\lambda g_{\nu\mu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\lambda\mu}) g_{\rho\nu} - \frac{1}{2} g^{\rho\mu} (\partial_\lambda g_{\mu\nu} + \partial_\nu g_{\lambda\mu} - \partial_\mu g_{\lambda\nu}) g_{\mu\rho}$$

which is indeed equal to zero.

11.3 The Riemann Normal Coordinate System

def. In a locally inertial frame (freely falling frame) the certain equality would hold

$$g_{\mu\nu}\Big|_{y=0} = \eta_{\mu\nu}$$

The fact that space is torsionless, we are able to say

$$\partial_\lambda g_{\mu\nu}\Big|_{y=0} = 0$$

thm. We now prove that a frame exists such that the Christoffel symbol is zero. Consider the geodesic equation

$$\ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0$$

We say that the initial conditions are given as

$$\begin{cases} x^\mu(\tau_0) = x^\mu_0 \\ \dot{x}^\mu(\tau_0) = V^\mu \end{cases}$$

The unique solution would be given as

$$x^\mu(\tau) = f^\mu(\tau, x_0, V)$$

where

$$\begin{cases} f^\mu(\tau_0, x_0, V) = x_0^\mu \\ \\ \dfrac{df^\mu}{d\tau}(\tau_0, x_0, V) = V^\mu \end{cases}$$

For a constant κ, we consider

$$\begin{cases} f^\mu(\tau, x_0, \kappa V) \\ \\ f^\mu(\kappa(\tau - \tau_0) + \tau_0, x_0, V) \end{cases}$$

which would both satisfy the equation as

$$\frac{df^\mu}{d\tau} = \kappa \frac{df^\mu(\tau', x_0, V)}{d\tau'}\Big|_{\tau' \to \kappa\tau}$$

$$\frac{d^2 f^\mu}{d\tau^2} = \kappa^2 \frac{d^2 f^\mu(\tau', x_0, V)}{d\tau^2}$$

now, consider a coordinate transformation from $x^\mu \to v^\mu$.

$$x^\mu(\tau) = f^\mu(\tau, x_0, V) = f^\mu(1, x_0, \tau v_0) = f^\mu(1, x_0, v(\tau))$$

In this frame, which we call the Riemann normal coordinate system,

$$v^\mu(\tau) = \tau V^\mu$$

and

$$\frac{d^2 v^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \dot{v}^\nu \dot{v}^\rho = 0$$

and thus the Christoffel symbol is zero.

12 Lecture 12

In this lecture, we delve into weights of tensors, which are how they scale of a power of the reciprocal of the Jacobian as they transform. Then, we see how we can calculate variations of a determinant of a function. By doing so, we notice that we need a stronger condition for which space-time is flat. We then derive the curvature by seeking a covariant tensor constructable with double derivatives of the metric.

12.1 Weights of Tensors

recall. Consider the following statement about the metric tensor that suggests that the metric is covariantly constant.

$$\nabla_\lambda g_{\mu\nu} = 0$$

Simply put, this face implies that the space is flat up to the first derivative of the metric. Also called metric compatibility, it implies that the covariant derivative is constructed in a way to keep the metric constant.

thm. On another note, now observe the transformation of the metric tensor

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x)$$

The metric's determinant can be seen to scale like the following.

$$g = \det(g(x)) \to \det(g'(x)) = g' = ||\frac{\partial x}{\partial x'}||^2 g$$

Note that we have defined g to represent the determinant of the metric. We thus claim the metric to be a tensor of weight 2, and its determinant to be a tensor density (not a tensor) of weight 2. The determinant of the negative square root of the metric is thus a scalar density with weight $w = 1$ (here, we define the weight to be the power of $||\partial x/\partial x'||$ not $||\partial x'/\partial x||$).

$$\sqrt{-g} \to \sqrt{-g'} = ||\frac{\partial x}{\partial x'}||\sqrt{-g}$$

12.2 Derivative of the Determinant and Divergence

thm. The determinant of a $n \times n$ matrix can be derivated as follows (the determinant's variation).

$$\begin{aligned}
||M|| &= \sum_{a_1,\cdots,a_n} \epsilon^{a_1\cdots a_n} M_{1a_1} M_{2a_2} \cdots M_{na_n} \\
\delta||M|| &= \sum_{a_1,\cdots,a_n} \epsilon^{a_1\cdots a_n} \Big(\sum_{j=1}^{n} M_{1a_1} \cdots \delta M_{ja_j} \cdots M_{na_n}\Big) \\
&= \sum_{a_1,\cdots,a_n} \epsilon^{a_1\cdots a_n} \Big(\sum_{j=1}^{n} M_{1a_1} \cdots \delta^b_{a_j} \cdots M_{na_n}\Big) \delta M_{jb} \\
&= \sum_{a_1,\cdots,a_n} \epsilon^{a_1\cdots a_n} \Big(\sum_{j=1}^{n} M_{1a_1} \cdots M_{ca_j} \cdots M_{na_n}\Big) (M^{-1})^{bc} \delta M_{jb} \\
&= \sum_{a_1,\cdots,a_n} \sum_{j=1}^{n} \Big(\epsilon^{a_1\cdots a_n} M_{1a_1} \cdots M_{ca_j} \cdots M_{na_n}\Big) (M^{-1})^{bc} \delta M_{jb} \\
&= \sum_{j=1}^{n} ||M|| \delta^j_c (M^{-1})^{bc} \delta M_{jb} = ||M|| (M^{-1})^{bc} \delta M_{cb}
\end{aligned}$$

thus we obtain

$$\delta||M|| = ||M||(M^{-1})^{ab}\delta M_{ba}$$

dividing both sides by the determinant,

$$\frac{\delta||M||}{||M||} = (M^{-1})^{ab}\delta M_{ba}$$

we can express the left side as the variation of the natural logarithm of the determinant, while the right hand side is a summation over all permutations of a and b. Doing so, we realize that it is equal to the trace of the matrix product $M^{-1}\delta M$.

$$\delta \ln||M|| = \mathrm{Tr}(M^{-1}\delta M)$$

thm. Now we return to the metric tensor. Derivating the determinant of the metric and the negative square root we derive that

$$\begin{aligned}\partial_\mu g &= g\; g^{\rho\sigma}\partial_\mu g_{\sigma\rho} \\ \partial_\mu\sqrt{-g} &= \frac{1}{2}\frac{1}{\sqrt{-g}}\partial_\mu g \\ &= \frac{1}{2}\sqrt{-g}\; g^{\rho\sigma}\partial_\mu g_{\sigma\rho}\end{aligned}$$

However, we know that

$$\begin{aligned}\Gamma^\lambda_{\mu\nu} &= \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \\ \Gamma^\lambda_{\mu\lambda} &= \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\rho\lambda} + \partial_\lambda g_{\mu\rho} - \partial_\rho g_{\mu\lambda})\end{aligned}$$

Thus we obtain the expressions

$$\begin{aligned}\partial_\mu\sqrt{-g} &= \sqrt{-g}\;\Gamma^\lambda_{\mu\lambda} \\ \partial_\mu g &= 2g\;\Gamma^\lambda_{\mu\lambda}\end{aligned}$$

In general, the covariant derivative of a tensor density becomes

$$\nabla_\lambda T^{\nu_1\cdots}_{\mu_1\cdots} = \partial_\lambda T^{\nu_1\cdots}_{\mu_1\cdots} - w\Gamma^\rho_{\lambda\rho}T^{\nu_1\cdots}_{\mu_1\cdots} + \left(\sum_j \Gamma^{\nu_j}_{\lambda\rho}T^{\nu_1\cdots\rho\cdots}_{\mu_1\cdots}\right) - \left(\sum_k \Gamma^\sigma_{\lambda\mu_k}T^{\nu_1\cdots}_{\mu_1\cdots\sigma\cdots}\right)$$

This is why the following hold

$$\begin{cases}\nabla_\mu g = 0 \\ \nabla_\mu\sqrt{-g} = 0\end{cases}$$

rmk. Take the current density. It can be easily noted how our definition lacked the notice of it actually being a tensor density.

$$
\begin{aligned}
J^\mu(x) \to J'^\mu(x) = ||\frac{\partial x}{\partial x'}||^{-1}\frac{\partial x'^\mu}{\partial x^\nu}J^\nu \\
\nabla_\mu J^\mu = \partial_\mu J^\mu = \partial_\mu J^\mu - \Gamma^\nu_{\mu\nu}J^\mu + \Gamma^\mu_{\mu\nu}J^\nu
\end{aligned}
$$

thm. A cool trick can be done for (1,0)-tensors of weight of weight zero or $w = 0$ (vectors). The divergence simply becomes

$$
\begin{aligned}
\nabla_\mu V^\mu &= \partial_\mu V^\mu + \Gamma^\mu_{\mu\rho}V^\rho \\
&= \partial_\mu V^\mu + \frac{\partial_\mu\sqrt{-g}}{\sqrt{-g}}V^\mu \\
&= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, V^\mu)
\end{aligned}
$$

def. If you observe the proper distance in terms of spherical coordinates,

$$
\begin{aligned}
ds^2 &= -dt^2 + dx^2 + dy^2 + dz^2 \\
&= -dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \\
&= g_{\mu\nu}dx^\mu dx^\nu
\end{aligned}
$$

using these coordinates, where $x^\mu = (t, r, \theta, \phi)$ rather than $x^\mu = (t, x, y, z)$, we notice that the metric changes from

$$
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

to

$$
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & r^2 & 0 \\
0 & 0 & 0 & r^2\sin^2\theta
\end{pmatrix}
$$

The metric in the upper case leads to $\Gamma^\lambda_{\mu\nu} = 0$ while the lower metric leads to $\Gamma^\lambda_{\mu\nu} \neq 0$ with $\sqrt{-g} = r^2\sin^2\theta$. We conclude that we need a more concrete condition for a certain space to be flat. The answer lies in the curvature of that space. Note that

$$
\begin{cases}
x = r\sin\theta\cos\phi \\
y = r\sin\theta\sin\phi \\
z = r\cos\theta \\
r = \sqrt{x^2 + y^2 + z^2}
\end{cases}
$$

12.3 Curvature and Riemann Curvature

note. We then ask the following question: can we construct a covariant tensor from $\partial_\lambda g_{\mu\nu}$? The answer is no! However, we can construct a covariant tensor from double derivatives of $g_{\mu\nu}$. The answer is curvature.

def. The curvature is defined as the commutator operator operated upon two covariant derivatives, or

$$[\nabla_\mu, \nabla_\nu]$$

we note that

$$[\partial_\mu, \partial_\nu]$$

We now calculate the double covariant derivative of an arbitrary vector.

$$\begin{aligned}
\nabla_\mu \nabla_\nu V^\lambda &= \partial_\mu(\nabla_\nu V^\lambda) - \Gamma^\rho_{\mu\nu}\nabla_\rho V^\lambda + \Gamma^\lambda_{\mu\rho}\nabla_\nu V^\rho \\
&= \partial_\mu(\partial_\nu V^\lambda + \Gamma^\lambda_{\nu\rho}V^\rho) - \Gamma^\rho_{\mu\nu}(\partial_\rho V^\lambda + \Gamma^\lambda_{\rho\sigma}V^\sigma) \\
&+ \Gamma^\lambda_{\mu\rho}(\partial_\nu V^\rho + \Gamma^\rho_{\nu\sigma}V^\sigma) \\
&= \partial_\mu\partial_\nu V^\lambda + \partial_\mu \Gamma^\lambda_{\nu\rho}V^\rho + \Gamma^\lambda_{\nu\rho}\partial_\mu V^\rho \\
&- \Gamma^\rho_{\mu\nu}(\partial_\rho V^\lambda + \Gamma^\lambda_{\rho\gamma}V^\sigma) + \Gamma^\lambda_{\mu\rho}(\partial_\nu V^\rho + \Gamma^\rho_{\nu\sigma}V^\sigma)
\end{aligned}$$

Considering only the remaining terms, we obtain

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu]V^\lambda &= (\partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\nu\rho})V^\rho + \left(\Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\nu\sigma} - \Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\mu\sigma}\right)V^\sigma \\
&= (\partial_\mu \Gamma^\lambda_{\nu\sigma} - \partial_\nu \Gamma^\lambda_{\mu\sigma} + \Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\nu\sigma} - \Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\mu\sigma})V^\sigma \\
&= R^\lambda_{\sigma\mu\nu}V^\sigma
\end{aligned}$$

We thus write the Riemann curvature as

$$R^\kappa_{\lambda\mu\nu} = \partial_\mu \Gamma^\kappa_{\nu\lambda} - \partial_\nu \Gamma^\kappa_{\mu\lambda} + \Gamma^\kappa_{\mu\rho}\Gamma^\rho_{\nu\lambda} - \Gamma^\kappa_{\nu\rho}\Gamma^\rho_{\mu\lambda}$$

where

$$[\nabla_\mu, \nabla_\nu]V^\lambda = R^\lambda_{\rho\mu\nu}V^\rho$$

In other notation, using $\Gamma^*_{\mu *}$

$$R^*_{*\mu\nu} = (\partial_\mu \Gamma_{\mu\nu} - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu])^*_*$$

curvature can alternatively defined as the field strength of curvature.

13 Lecture 13

We investigate important properties of the curvature tensor and introduce the Bianchi identity.

def. The levi-civita symbol is defined as

$$\epsilon^{a_1 a_2 \cdots a_n} = \begin{cases} +1 & \text{even permutation} \\ -1 & \text{odd permutation} \\ 0 & \text{otherwise} \end{cases}$$

def. The determinant is defined through this symbol as

$$||M|| = \sum_{a_1,\cdots,a_n} \epsilon^{a_1\cdots a_n} M_{1a_1} \cdots M_{na_n}$$

recall. The important theorem from the last lecture was

$$\delta \ \ln||M|| = \mathrm{Tr}(M^{-1}\delta M)$$

and from this, we have the following corollary

$$\partial_\mu ||g|| = 2\ \Gamma^\nu_{\mu\nu} ||g||$$

with $\nabla_\mu ||g|| = 0$.

thm. From the above, we theorize that tensors with different weights transform like the following. As a density,

$$\partial_\mu T \to \partial'_\mu T' = \partial'_\mu T = \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu T$$

but with a weight w,

$$\begin{aligned} \partial_\mu T \to \partial'_\mu T' &= \partial'_\mu \Big(||\frac{\partial x'}{\partial x}||^{-w} T \Big) \\ &= ||\frac{\partial x'}{\partial x}||^{-w} \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu T - w ||\frac{\partial x'}{\partial x}||^{-w} \frac{\partial x^\lambda}{\partial x'^\rho} \frac{\partial^2 x'^\rho}{\partial x^\mu \partial x^\lambda} T \end{aligned}$$

(notes on weight redacted)

13.1 Properties of Curvature

recall. We learnt that curvature is defined as

$$[\nabla_\mu, \nabla_\nu] V^\lambda = R^\lambda_{\rho\mu\nu} V^\rho$$

where

$$\begin{aligned} R^\lambda_{\rho\mu\nu} &= (\partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu])^\lambda_\rho \\ &= \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\rho} - \Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma} \Gamma^\sigma_{\mu\rho} \end{aligned}$$

What about of the general case?

$$[\nabla_\mu, \nabla_\nu] T^{\lambda_1 \cdots \lambda_p}_{\kappa_1 \cdots \kappa_q}$$

for now we first note that for vectors with lower indices,

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu] V_\lambda &= \nabla_\mu \nabla_\nu V_\lambda - (\mu \leftrightarrow \nu) \\
&= \partial_\mu (\nabla_\nu V_\lambda) - \Gamma^\rho_{\mu\nu} \nabla_\rho V_\lambda - \Gamma^\rho_{\mu\nu} \nabla_\nu V_\rho - (\mu \leftrightarrow \nu) \\
&= \partial_\mu (\partial_\nu V_\lambda - \Gamma^\rho_{\nu\lambda} V_\rho) - \Gamma^\rho_{\mu\lambda} (\partial_\nu V_\rho - \Gamma^\sigma_{\nu\rho} V_\sigma) - (\mu \leftrightarrow \nu) \\
&= -R^\rho_{\lambda\mu\nu} V_\rho
\end{aligned}$$

recall. Given the following way that the Christoffel symbol transforms, we can see that the curvature transforms covariantly.

$$\Gamma^\lambda_{\mu\nu} \to \Gamma'^\lambda_{\mu\nu} = \frac{\partial x'^\lambda}{\partial x^\kappa} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \Gamma^\kappa_{\rho\sigma} + \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}$$

this is left as homework, to check that,

$$R^\lambda_{\rho\mu\nu} \to R'^\lambda_{\rho\mu\nu} = \frac{\partial x'^\lambda}{\partial x^\kappa} \frac{\partial x^\sigma}{\partial x'^\rho} \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} R^\kappa_{\sigma\alpha\beta}$$

def. The object $R^\lambda_{\rho\mu\nu}$ is called the Riemann curvature. There are many characteristics that are important regarding the Riemann curvature. Suppose lowering the indices,

$$R_{\kappa\lambda\mu\nu} = g_{\kappa\rho} R^\rho_{\lambda\mu\nu}$$
$$R_{\kappa\lambda\mu\nu} = -R_{\kappa\lambda\nu\mu} = -R_{\lambda\kappa\mu\nu} = R_{\lambda\kappa\nu\mu} = R_{\mu\nu\kappa\lambda}$$

rmk. A trick regarding obtaining the relationship above, in the local inertial frame, $\Gamma^\lambda_{\mu\nu} = 0$ and $\partial_\mu g_{\nu\rho} = 0$. The curvature becomes,

$$\begin{aligned}
R_{\lambda\rho\mu\nu} &= \partial_\mu (g_{\lambda\alpha} \Gamma^\alpha_{\nu\rho}) - \partial_\nu (g_{\lambda\alpha} \Gamma^\alpha_{\mu\rho}) \\
&= \frac{1}{2} \partial_\mu (\partial_\nu g_{\lambda\rho} + \partial_\rho g_{\nu\lambda} - \partial_\lambda g_{\nu\rho} - \frac{1}{2} \partial_\nu (\partial_\mu g_{\lambda\rho} + \partial_\rho g_{\mu\lambda} - \partial_\lambda g_{\mu\rho}) \\
&= \frac{1}{2} (\partial_\mu \partial_\nu g_{\lambda\rho} + \partial_\mu \partial_\rho g_{\nu\lambda} - \partial_\mu \partial_\lambda g_{\nu\rho} - \partial_\nu \partial_\mu g_{\lambda\rho} - \partial_\nu \partial_\rho g_{\mu\lambda} + \partial_\nu \partial_\lambda g_{\mu\rho}) \\
&= \frac{1}{2} (\partial_\mu \partial_\rho g_{\nu\lambda} - \partial_\mu \partial_\lambda g_{\nu\rho} - \partial_\nu \partial_\rho g_{\mu\lambda} + \partial_\nu \partial_\lambda g_{\mu\rho})
\end{aligned}$$

13.2 The Bianchi Identity

def. Any commutator satisfies the Bianchi identity,

$$[A[B, C]] + [B[C, A]] + [C[A, B]] = 0$$

As a commutative operator, it also works for the covariant derivative, and

$$
\begin{aligned}
0 &= [\nabla_\lambda[\nabla_\mu, \nabla_\nu]]V^\kappa + (\text{cyclic } \lambda, \mu, \nu) \\
&= \nabla_\lambda([\nabla_\mu, \nabla_\nu]V^\kappa) - [\nabla_\mu, \nabla_\nu](\nabla_\lambda V^\kappa) + (\text{cyclic } \lambda, \mu, \nu) \\
&= \nabla_\lambda(R^\kappa_{\rho\mu\nu} V^\rho) - (R^\kappa_{\rho\mu\nu}\nabla_\lambda V^\rho - R^\rho_{\lambda\mu\nu}\nabla_\rho V^\kappa) + (\text{cyclic } \lambda, \mu, \nu) \\
&= \nabla_\lambda R^\kappa_{\rho\mu\nu} V^\rho + R^\kappa_{\rho\mu\nu}\nabla_\lambda V^\rho - R^\kappa_{\rho\mu\nu}\nabla_\lambda V^\rho - R^\rho_{\lambda\mu\nu}\nabla_\rho V^\kappa + (\text{cyclic } \lambda, \mu, \nu) \\
&= (\nabla_\lambda R^\kappa_{\rho\mu\nu} + \nabla_\mu R^\kappa_{\rho\nu\lambda} + \nabla_\nu R^\kappa_{\rho\lambda\mu})V^\rho - (R^\rho_{\lambda\mu\nu} + R^\rho_{\mu\nu\lambda} + R^\rho_{\nu\lambda\mu})\nabla_\rho V^\kappa = 0
\end{aligned}
$$

in result, we obtain the differential Bianchi identity,

$$\nabla_\lambda R^{\rho\sigma}_{\mu\nu} + \nabla_\mu R^{\rho\sigma}_{\nu\lambda} + \nabla_\nu R^{\rho\sigma}_{\lambda\mu} = 0$$

also,

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} = 0$$

thm. If and only if $R^\kappa_{\lambda\mu\nu} = 0$, one can find a coordinate system where $g_{\mu\nu} = \eta_{\mu\nu}$, i.e., flat space-time.

14 Lecture 14

In this lecture, we learn the commutator and permutator operator and relevant notation. Afterwards, through the Bianchi identity, we derive the Einstein curvature, arriving at the Einstein field equations. As a constituent of the equations, we investigate the energy momentum tensor for electrodynamics.

recall. In the last class, we defined the Riemann curvature as

$$
\begin{aligned}
R^\kappa_{\lambda\mu\nu} =& \partial_\mu \Gamma^\kappa_{\nu\lambda} - \partial_\nu \Gamma^\kappa_{\mu\lambda} + \Gamma^\kappa_{\mu\rho}\Gamma^\rho_{\nu\lambda} - \Gamma^\kappa_{\nu\rho}\Gamma^\rho_{\mu\lambda} \\
R_{\kappa\lambda\mu\nu} =& R_{\mu\nu\kappa\lambda} = R_{[\kappa,\lambda][\mu,\nu]} \\
\nabla_{[\lambda} R_{\mu,\nu]\rho\sigma} =& 0
\end{aligned}
$$

14.1 Commutator and Permutator Notation

def. With the commutator notation in the lower indices, we write (for tensor $T_{\lambda_1\lambda_2\cdots\lambda_n}$),

$$T_{[\lambda_1\lambda_2\cdots\lambda_n]} = \sum_\sigma \frac{1}{n!}\text{sgn}(\sigma) T_{\lambda_{\sigma(1)})\lambda_{\sigma(2)}\cdots\lambda_{\sigma(n)}}$$

Where σ denotes a permutation for a totally anti-symmetric tensor. For a totally symmetric tensor, we obtain

$$T_{(\lambda_1\lambda_2\cdots\lambda_n)} = \sum_\sigma \frac{1}{n!} T_{\lambda_{\sigma(1)}\lambda_{\sigma(2)}\cdots\lambda_{\sigma(n)}}$$

thm. Using this notation, we can rewrite tensors like the following.

$$\begin{aligned}
F_{\mu\nu} =& \partial_\mu A_\nu - \partial_\nu A_\mu \\
M_{(\mu\nu)} =& \frac{1}{2} M(M_{\mu\nu} + M_{\nu\mu}) \\
F_{[\mu\nu]} &= F_{\mu\nu} \\
M_{[\mu\nu]} &= \frac{1}{2}(M_{\mu\nu} - M_{\nu\mu}) \\
\partial_{[\lambda} F_{\mu\nu]} &= 0 \\
g_{\mu\nu} &= g_{(\mu\nu)} \\
&= \frac{1}{2}(\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} \\
g_{[\mu\nu]} &= 0 \\
&= \frac{1}{6}(\partial_\lambda F_{\mu\nu} - oar)_\lambda F_{\nu\mu} + \cdots) \\
M_{\mu\nu} &= M_{[\mu\nu]} + M_{(\mu\nu)} \\
\epsilon^{\kappa\lambda\mu\nu} &= \epsilon^{[\kappa\lambda\mu\nu]}
\end{aligned}$$

14.2 Ricci Curvature

def. We now use metric contraction to create a expression with 2 indices. We state that there is only one way to contract two indices from the expression above, and we call the result the Ricci curvature.

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} = R^\lambda_{\mu\nu\lambda} = R_{\nu\mu}$$

The Ricci curvature can be again contracted by the metric to give a scalar curvature.

$$R = g^{\mu\nu} R_{\mu\nu} = R^{\mu\nu}_{\mu\nu}$$

Note that the order we went through was Riemann, Ricci, and scalar curvatures. In low dimensions, these are identical.

recall. The Bianchi identity was given as

$$\nabla_\lambda R_{\mu\nu\rho\sigma} + \lambda_\mu R_{\nu\lambda\rho\sigma} + \nabla_\nu R_{\lambda\mu\rho\sigma} = 0$$

When contracting each term with the metric, we can only choose one from the first three and one from the latter two. Using $g^{\lambda\rho}$ to contract,

$$\nabla_\lambda R^\lambda_{\mu\nu\sigma} - \nabla_\mu R_{\nu\sigma} + \nabla_\nu R_{\mu\sigma} = 0$$

then,

$$\nabla_\lambda R^\lambda_{\sigma\mu\nu} - \nabla_\mu R_{\nu\sigma} - \nabla_\nu R_{\mu\sigma} = 0$$

select one from μ and ν, and contract with sigma ($g^{\sigma\mu}$) to obtain

$$-\nabla_\lambda R^\lambda_\nu - \nabla_\mu R^\mu_\nu + \nabla_\nu R = 0$$

notice that the above becomes

$$\begin{aligned}
-2\nabla_\lambda R^\lambda_\nu + \nabla_\nu R =& \nabla_\lambda R^\lambda_\nu - \frac{1}{2}\nabla_\nu R = 0 \\
\nabla_\lambda (R^\lambda_\nu - \frac{1}{2}\delta^\lambda_\nu R) =& 0 \\
\nabla_\lambda (R^{\lambda\mu} - \frac{1}{2}g^{\lambda\mu} R) = 0 \\
\nabla^\lambda (R_{\lambda\mu} - \frac{1}{2}g_{\lambda\mu} R) = 0
\end{aligned}$$

14.3 Einstein Curvature and the Einstein Field Equations

def. We newly define the Einstein curvature/tensor as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R$$

thus obtaining

$$\nabla_\mu G^\mu_\nu = 0$$

which is covariantly conserved.

note. we use for the above,

$$g^{\lambda\rho}\nabla_\mu R_{\nu\lambda\rho\sigma} = \nabla_\mu (g^{\lambda\rho} R_{\nu\lambda\rho\sigma})$$

thm. In Newtonian gravity,

$$\nabla^2\Phi = 4\pi G\rho$$

The Einstein field equations are

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

The left is the Einstein curvature and the right-side is the energy-momentum tensor. Note that

$$\nabla_\mu G^\mu_\nu = 0$$

And G^μ_ν is identically conserved/off-shell conserved and

$$\nabla_\mu T^\mu_\nu = 0$$

is on-shell conserved. The left-hand-side of the equation denotes information about space-time, while the right-hand-side writes information about matter.

14.4 The Energy-momentum Tensor in Electrodynamics

rmk. The energy-momentum tensor in electrodynamics can be expressed as

$$T^{\mu\nu} = F^{\mu\rho}F_{\rho}^{\nu} - \frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}$$

we note that

$$T_{\mu}^{\mu} = 0$$

and that the tensor is traceless.

$$\begin{aligned}
\nabla_{\mu}T^{\mu\nu} =& \nabla_{\mu}(F^{\mu\rho}F_{\rho}^{\nu} - \frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}) \\
=& \nabla_{\mu}F^{\mu\rho}F_{\rho}^{\nu} + F^{\mu\rho}\nabla_{\mu}F_{\rho}^{\nu} - \frac{1}{4}\nabla^{\nu}(F^{\rho\sigma}F_{\rho\sigma}) \\
=& \nabla_{\mu}F^{\mu\rho}F_{\rho}^{\nu} + F_{\mu\rho}\nabla^{\mu}F^{\nu\rho} - \frac{1}{2}\nabla^{\nu}F^{\rho\sigma}F_{\rho\sigma} \\
=& \nabla_{\mu}F^{\mu\rho}F_{\rho}^{\nu} + \frac{1}{2}F_{\mu\rho}(\nabla^{\mu}F^{\nu\rho} - \nabla^{\rho}F^{\nu\mu} - \nabla^{\nu}F^{\mu\rho}) \\
=& \nabla_{\mu}F^{\mu\rho}F_{\rho}^{\nu} + \frac{3}{2}F_{\mu\rho}\nabla^{[\rho}F^{\mu\nu]}
\end{aligned}$$

now,

$$\begin{aligned}
\nabla_{\mu}T^{\mu\nu} =& \nabla_{\mu}F^{\mu\rho}F_{\rho}^{\nu} + \frac{3}{2}F_{\mu\rho}\nabla^{[\rho}F^{\mu\nu]} \\
=& J^{\rho}F_{\rho}^{\nu} + 0 \\
=& F_{\rho}^{\nu}J^{\rho}
\end{aligned}$$

with $J^{\mu} = 0$, $\nabla_{\mu}T^{\mu\nu} = 0$.

note. Note that we used

$$F_{\mu\rho}\nabla^{\mu}F^{\nu\rho} = -F_{\rho\mu}\nabla^{\mu}F^{\nu\rho} = -F_{\mu\rho}\nabla^{\rho}F^{\nu\mu}$$

rmk. next class, we derive that with the existence of a particle,

$$\nabla_{\mu}T^{\mu\nu} = -F_{\rho}^{\nu}J^{\rho}$$

summing up with the expression above to become zero.

$$T_{\text{total}}^{\mu\nu} = T_{\text{EM}}^{\mu\nu} + T_{\text{particle}}^{\mu\nu}$$

and

$$\nabla_{\mu}T_{\text{total}}^{\mu\nu} = 0$$

15 Lecture 15

In this lecture, we finish off the discussion about the energy-momentum tensor in electrodynamics, calculating its divergence to derive energy and momentum conservation.

15.1 More on the Energy-momentum Tensor In Electrodynamics

recall. We recall a few things from last class. We first learned that the Einstein tensor was given as

$$G_\mu = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

forming the Einstein field equation,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

The Einstein curvature satisfies the off-shell (Bianchi identity), irrelevant of coordinate system

$$\nabla_\mu G^{\mu\nu} = 0$$

whereas the energy-stress tensor satisfies the on-shell identity

$$\nabla_\mu T^{\mu\nu} = 0$$

we also specified the stress energy tensor for electromagnetic fields where

$$T^{\mu\nu}_{\text{EM}} = F^{\mu\rho} F^{\nu}_{\rho} - \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}$$

rmk. For the following equation to have same weights,

$$\nabla_\mu J^{\mu\nu} - J^\nu = 0$$

we require a little change in our definition of current density, where you add the square root of the metric in the denominator.

thm. We now slightly state the Lagrangian formalism of general relativity. We integrate (find the variation of)

$$\int \sqrt{-g} R + \mathcal{L}_{\text{matter}} + m\sqrt{-g_{\mu\nu} \dot{x}^2 \dot{x}^2} \, d^4x$$

The first term gives $16\pi G_{\mu\nu}$ and the second $\sqrt{-g} F_{\mu\nu} F^{\mu\nu}$.

def. The energy momentum tensor for a point particle is

$$T^{\mu\nu}(x) = \sum_n \int_{-\infty}^{\infty} m_n \frac{dx_n^\mu}{d\tau} \frac{dx_n^\nu}{d\tau} \frac{\delta^{(4)}(x - x_n(\tau_n))}{\sqrt{-g(x)}} \, d\tau$$

thm. We now calculate the divergence of this tensor. We obtain

$$\nabla_\mu T^{\mu\nu}(x) = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\rho} T^{\rho\nu} + \Gamma^\nu_{\mu\rho} + \Gamma^\nu_{\mu\rho} T^{\mu\rho}$$

and

$$\begin{aligned}\sum_n \partial_\mu \int m\dot{x}^\mu_n \dot{x}^\nu_n \frac{\delta^{(4)}(x-x_n)}{\sqrt{-g(x)}}\, d\tau &= \sum_n \int m\dot{x}^\mu_n \dot{x}^\nu_n \partial_\mu \Big(\frac{\delta^{(4)}(x-x_n)}{\sqrt{-g}}\Big)\, d\tau \\ &= \sum_n \int m\dot{x}^\nu_n \dot{x}^\mu_n \Big[\frac{\partial_\mu \delta^{(4)}(x-x_n)}{\sqrt{-g}} + \partial_\mu\Big(\frac{1}{\sqrt{-g}}\Big)\delta^{(4)}(x-x_n)\Big]\, d\tau \\ &= \sum_n \int m\dot{x}^\nu_n \dot{x}^\mu_n \Big(\frac{-\partial/\partial x^\mu_n \delta^{(4)}(x-x_n)}{\sqrt{-g}} - \Gamma^\lambda_{\lambda\mu}\frac{\delta^{(4)}(x-x_n)}{\sqrt{-g}}\Big)\, d\tau \\ &= \sum_n \int m - \dot{x}^\nu_n \frac{d/dt\delta^{(4)}(x-x_n)}{\sqrt{-g}} - \Gamma^\lambda_{\lambda\mu} T^{\mu\nu}\, d\tau\end{aligned}$$

and thus the divergence becomes

$$\nabla_\mu T^{\mu\nu}(x) = \sum_n m \int -\frac{\dot{x}^\nu_n(\tau) d/dt\delta^{(4)}(x-x_n(\tau))}{\sqrt{-g(x)}} + \Gamma^\nu_{\mu\rho} T^{\mu\rho}\, d\tau$$

using integration by parts,

$$= \sum_n m \int \frac{\ddot{x}^\nu_n(\tau)\delta^{(4)}(x-x_n(\tau))}{\sqrt{-g}} + \frac{\Gamma^\nu_{\mu\rho}\dot{x}^\mu \dot{x}^\rho \delta^{(4)}(x-x_n)}{\sqrt{-g}}\, d\tau$$

we arrive at

$$\sum_n m \int (\ddot{x}^\nu_n + \Gamma^\nu_{\mu\rho}\dot{x}^\mu_n \dot{x}^\rho_n)\frac{\delta^{(4)}(x-x_n)}{\sqrt{-g}}\, d\tau$$

which shall be equal the Lorentz force

$$= \sum_n \int qF^\nu_\rho \dot{x}^\rho_n \frac{\delta^{(4)}(x-x_n)}{\sqrt{-g}}; d\tau = F^\nu_\rho(x) J^\rho(x)$$

as

$$m\frac{D^2 x^\mu}{D\tau^2} = m(\ddot{x}^\mu + \Gamma^\mu_{\rho\sigma}\dot{x}^\rho \dot{x}^\sigma) = qF^\mu_\nu \dot{x}^\nu$$

now,

$$\begin{aligned}\nabla_\mu T^{\mu\nu}_{\text{EM}} &= -F^\nu_\rho J^\rho \\ \nabla_\mu (T^{\mu\nu}_{\text{particle}} + T^{\mu\nu}_{\text{EM}}) &= 0\end{aligned}$$

now this encapsulates both energy and momentum conservation.

note. for the metric tensor,

$$\begin{aligned}
0 =& \nabla_\mu \sqrt{-g} = \partial_\mu \sqrt{-g} - \Gamma^\lambda_{\lambda\mu}\sqrt{-g} \\
\partial_\mu \sqrt{-g} =& \Gamma^\lambda_{\lambda\mu} \\
\partial_\mu \frac{1}{\sqrt{-g}} =& -\frac{\partial_\mu \sqrt{-g}}{(\sqrt{-g})^2} = \frac{-\Gamma^\lambda_{\lambda\mu}}{\sqrt{-g}} \\
\partial_\mu (\sqrt{-g})^n =& n\Gamma^\lambda_{\lambda\mu}(\sqrt{-g})^n
\end{aligned}$$

16 Lecture 16

By approximating the metric as a slight variation of a flat metric, we obtained a linearized version of the Einstein field equations Then, we get a elementary introduction to gauge symmetry.

16.1 The Linearized Einstein Field Equations

recall. The Einstein field equations were given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^2} T_{\mu\nu}$$

whereas for a point particle, the stress-energy tensor was given as

$$T^{\mu\nu} = \sum_b m_n \int \dot{x}^\mu_n \dot{x}^\nu_n \frac{\delta^{(4))}(x - x_n(\tau))}{\sqrt{-g(x)}}\, d\tau$$

the $(0, 0)$ indice of the stress energy tensor is

$$T^{00} \approx m_n c^2 \dot{t}^2$$

on the Earth's surface,

$$g_{\mu\nu} \approx \eta_{\mu\nu} \pm 10^{-6}$$

rmk. We use the perturbation method to obtain a linearised version of the Einstein field equations. By linearization we mean that we express the metric as

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$$

also expressible as $\delta g_{\mu\nu} = h_{\mu\nu}$. What about the inverse of the metric?

$$g^{\mu\nu} \approx \eta^{\mu\nu} - \eta^{\mu\rho} h_{\rho\sigma} \eta^{\sigma\nu} = \eta^{\mu\nu} + \delta g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$$

as we require that they multiply to give the Kronecker delta. It can also be described by $\delta(M^{-1}) = -(M)^{-1}\delta M (M)^{-1}$. The Christoffel symbol, on the other hand, will be transformed accordingly

$$\begin{aligned}\Gamma^{\lambda}_{\mu\nu} &= \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \\ &= \frac{1}{2}(\eta^{\lambda\rho} - h^{\lambda\rho})(\partial_\mu h_{\rho\nu} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}) \\ &= \frac{1}{2}(\partial_\mu h^{\lambda}_{\nu} + \partial_\nu h^{\lambda}_{\mu} - \partial^\lambda h_{\mu\nu})\end{aligned}$$

Let's try and do something similar for the Ricci tensor.

$$\begin{aligned}R^{\kappa}_{\lambda\mu\nu} =& \partial_\mu \Gamma^{\kappa}_{\nu\lambda} - \partial_\nu \Gamma^{\kappa}_{\mu\lambda} + \Gamma^{\kappa}_{\mu\rho}\Gamma^{\rho}_{\nu\lambda} - \Gamma^{\kappa}_{\nu\rho}\Gamma^{\rho}_{\mu\lambda} \\ =& \frac{1}{2}\partial_\mu(\partial_\nu h^{\kappa}_{\lambda} + \partial_\lambda h^{\kappa}_{\nu} - \partial^\kappa h_{\nu\lambda}) - \frac{1}{2}\partial_\nu(\partial_\mu h^{\kappa}_{\lambda} + \partial_\lambda h^{\kappa}_{\mu} - \partial^\kappa h_{\mu\lambda}) \\ =& \frac{1}{2}\partial_\mu\partial_\lambda h^{\kappa}_{\nu} - \frac{1}{2}\partial_\mu\partial^\kappa h_{\nu\lambda} - \frac{1}{2}\partial_\nu\partial_\lambda h^{\kappa}_{\mu} + \frac{1}{2}\partial_\mu\partial^\kappa h_{\mu\lambda} \\ R_{\lambda\nu} \approx& \frac{1}{2}\partial_\lambda\partial_\mu h^{\mu}_{\nu} - \frac{1}{2}\partial_\mu\partial^\mu h_{\nu\lambda} - \frac{1}{2}\partial_\lambda\partial_\nu h^{\mu}_{\mu} + \frac{1}{2}\partial_\nu\partial_\mu h^{\mu}_{\lambda} \\ =& \partial_\mu\partial_{(\lambda} h^{\mu}_{\nu)} - \frac{1}{2}\Box h_{\nu\lambda} - \frac{1}{2}\partial_\nu\partial_\lambda h^{\rho}_{\rho} \\ R \approx& \eta^{\lambda\nu}R_{\lambda\nu} \approx -\Box h^{\lambda}_{\lambda} + \partial_\mu\partial_\nu h^{\mu\nu}\end{aligned}$$

where $\Box = \partial_\mu\partial^\mu$. Therefore the linearized Einstein field equation becomes

$$G_{\mu\nu} \approx R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R$$

thm. Another way of expressing this statement is by contracting the field equations by $g^{\mu\nu}$. We obtain

$$R - \frac{1}{2}DR = \frac{(2-D)}{2}R = 8\pi G T^{\mu}_{\mu}$$

assuming that $D \neq 2$,

$$R = \frac{16\pi G}{2-D}R^{\mu}_{\mu}$$

and

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} + \frac{g_{\mu\nu}}{2-D}T^{\lambda}_{\lambda})$$

16.2 Gauge Symmetry

def. General covariance is a central principle of general relativity. It states that for an arbitrary coordinate transformation is always possible (Gauge symmetry). Note that coordinate trnasformations are also called diffeomorphisms.

note. We later learn that if we utilize the gauge choice, we can obtain

$$\partial_\mu h^{\mu}_{\nu} - \frac{1}{4}\partial_\nu h^{\mu}_{\mu} = 0$$

simplifying the equation above as

$$R_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu}$$

with $T_{\mu\nu} = 0$ in vacuum, the Einstein field equations predict gravitational waves by

$$\Box h_{\mu\nu} = 0$$

def. There are two perspectives regarding general coordinate transformations. The first is changing the coordinate system,

$$\begin{cases} x^\mu \to x'^\mu(x) \\ \partial_\mu \to \partial'_\mu = \dfrac{\partial x^\nu}{\partial x'^\mu}\partial_\nu \\ \Phi(x) \to \Phi'(x') = \Phi(x) \end{cases}$$

which is a passive transformation. There are also active transformations, where

$$\begin{cases} x^\mu \to x^\mu \\ \partial_\mu \to \partial_\mu \\ \Phi(x) \to \Phi'(x) = \Phi(x'(x)) \end{cases}$$

17 Lecture 17

By looking into how transformations would transform under active transformations, we arrive at the Lie derivative which is another covariant version of the derivative.

17.1 The Lie Derivative

recall. Last class, we mentioned the two sides of a diffeomorphism (a coordinate transformation that is both ways differentiable), passive and active. Consider the following transformation $x^\mu \to x'^\mu(x)$. A tensor would transform like the following

$$T^\mu_\nu(x) \to T'^\mu_\nu(x') = ||\frac{\partial x}{\partial x'}||^w \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\nu}T^\rho_\sigma(x)$$

This would be called the passive aspect of a diffeomorphism. An active aspect of a diffeomorphism would be when an actual event would change its coordinates. The partial derivatives would remain the same, as we would be using the same coordinate system. The tensor would transform as follows:

$$T^\mu_\nu(x) \to T'^\mu_\nu(x) = ||\frac{\partial x'}{\partial x}||^w \frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x'^\sigma}{\partial x^\nu}T^\rho_\sigma(x')$$

ex. Consider the following scalar transformation, $\phi(x) \to \phi'(x) = \phi(x')$.

def. Infinitesimal transformation where $f^{\mu}_{s=0}(x) = x^{\mu}$ and

$$\xi^{\mu}(x) = \frac{df^{\mu}_{s}(x)}{ds}|_{s=0} = \delta x^{\mu}$$

now consider the transformation

$$\begin{aligned} x^{\mu} \to x'^{\mu} &= f^{\mu}_{s}(x) = x^{\mu} + s\frac{df^{\mu}}{ds}|_{s=0} + \mathcal{O}(s^2) = x^{\mu} + s\xi^{\mu}(x) + \theta(s^2) \\ x^{\mu} \to x'^{\mu} &\approx x^{\mu} + \xi^{\mu}(x) = x^{\mu} + \delta x^{\mu} \end{aligned}$$

In this process, how much would the transformation themselves transform?

$$\begin{aligned} \frac{\partial x'^{\mu}}{\partial x^{\nu}} \approx& \frac{\partial}{\partial x^{\nu}}(x^{\mu} + \xi^{\mu}) = \delta^{\mu}_{\nu} + \partial_{\nu}\xi^{\mu} \\ \frac{\partial x^{\mu}}{\partial x'^{\nu}} \approx& \frac{\partial}{\partial x'^{\nu}}(x'^{\mu} - \xi^{\mu}(x')) = \delta^{\mu}_{\nu} - \partial_{\nu}\xi^{\mu} \end{aligned}$$

thus

$$\begin{cases} \delta\left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right) = \partial_{\nu}\xi^{\mu} \\ \\ \delta\left(\frac{\partial x^{\mu}}{\partial x'^{\nu}}\right) = -\partial_{\nu}\xi^{\mu} \end{cases}$$

note. we used

$$\begin{aligned} x' \approx& x + \xi(x) \\ x \approx& x' - \xi(x) = x' - \xi(x') \\ \xi(x') \approx& \xi(x + \xi) = \xi(x) + \xi^{\lambda}\partial_{\lambda}\xi \end{aligned}$$

We conclude that in passive diffeomorphisms, $\delta\phi = 0$ and for active diffeomorphisms, $\delta\phi = \xi^{\mu}\partial_{\mu}\phi(x)$, using $\phi(x + \xi) = \phi(x) + \xi^{\mu}\partial_{\mu}\phi(x)$.

thm. What is the variation of the determinant? Note that $\delta||M|| = ||M||\mathrm{Tr}(M^{-1}\delta M)$.

$$\delta||\frac{\partial x'}{\partial x}|| = ||\frac{\partial x'}{\partial x}|| = ||\frac{\partial x'}{\partial x}||\frac{\partial x^{\mu}}{\partial x'^{\nu}}\delta\left(\frac{\partial x'^{\nu}}{\partial x^{\mu}}\right)|_{s=0} = \partial_{\mu}\xi^{\mu}$$

thus for a whole tensor, the passive variation becomes

$$\delta_{\text{passive}}T^{\mu}_{\nu} = -w\partial_{\lambda}\xi^{\lambda}T^{\mu}_{\nu} + \partial_{\rho}\xi^{\mu}T^{\rho}_{\nu} - \partial_{\nu}\xi^{\rho}T^{\mu}_{\rho}$$

while the active variation becomes

$$\delta_{\text{active}}T^{\mu}_{\nu} = +w\partial_{\lambda}\xi^{\lambda}T^{\mu}_{\nu} - \partial_{\rho}\xi^{\mu}T^{\rho}_{\nu} + \partial_{\nu}\xi^{\rho}T^{\mu}_{\rho} + \xi^{\rho}\partial_{\rho}T^{\mu}_{\nu}$$

The last term of the active variation is called the transport term while the second and third term together are called the angular term. The first term is simply the weight term. We also denote this variation as the Lie derivative

$$\mathcal{L}_\xi T^\mu_\nu$$

The derivative can be generalized like the following

$$\mathcal{L}_\xi T^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_q} = \xi^\rho \partial_\rho T^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_q} - \sum_{i=1}^{p} \partial_\rho \xi^{\mu_i} T^{\cdots\rho\cdots}_{\nu_1\cdots\nu_q} + \sum_{j=1}^{q} \partial_{\nu_j} \xi^\rho T^{\mu_1\cdots\mu_p}_{\cdots\rho\cdots} + w\partial_\rho \xi^\rho T^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_q}$$

A surprising fact is that the derivative is covariant, and the above equates to

$$= \xi^\rho \nabla_\rho T^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_q} - \sum_{i=1}^{p} \nabla_\rho \xi^{\mu_i} T^{\cdots\rho\cdots}_{\nu_1\cdots\nu_q} + \sum_{j=1}^{q} \nabla_{v_j} \xi^\rho T^{\mu_1\cdots\mu_p}_{\cdots\rho\cdots} + w\nabla_\rho \xi^\rho T^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_q}$$

If insert Christoffel symbols, they all cancel out. Observe that

$$\begin{aligned}
&= \text{first row} + \sum_{i=1}^{p} \left(\xi^\rho \Gamma^{\mu_i}_{\rho\sigma} - \Gamma^{\mu_i}_{\sigma\rho}\xi^\rho\right) T^{\cdots\sigma\cdots}_{\nu_1\cdots\nu_q} - \sum_{j=1}^{q} \left(\xi^\rho T^{\sigma}_{\rho\nu_j} - \Gamma^{\sigma}_{\nu_j\rho}\xi^\rho\right) T^{\mu_1\cdots}_{\sigma\cdots} \\
&\quad - w\xi^\rho \Gamma^\sigma_{\rho\sigma} T^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_q} + w\Gamma^\rho_{\rho\sigma}\xi^\sigma T^{\mu_1\cdots}_{\nu_1\cdots}
\end{aligned}$$

rmk. An important property of the Lie derivative is that it satisfies the Leibniz rule,

$$\mathcal{L}_\xi(TS) = (\mathcal{L}_\xi T)S + T(\mathcal{L}_\xi S)$$

The Lagrangian of a particle is given as

$$\mathcal{L}(F_{\mu\nu}, g_{\mu\nu}) = -\frac{1}{4}\sqrt{-g}g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} = -\frac{1}{4}F^2$$

it is a scalar density, with

$$\delta_{\text{active}}\mathcal{L} = \mathcal{L}_\xi \mathcal{L} = \partial_\mu(\xi^n \mathcal{L})$$

prop. Consider the Lagrangian of a particle,

$$\mathcal{L}_{\text{particle}} = -m\sqrt{-g_{\mu\nu}(x(\tau))\dot{x}^\mu(\tau)\dot{x}^\nu(\tau)}$$

the infinitesimal displacement becomes

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$$

18 Lecture 18

We investigate the Killing equation, which is an equation that seeks vectors such that a *metric's Lie derivative is zero. We discover these vectors to imply symmetries within space-time.*

18.1 The Killing Equation

recall. Last class we have learned the Lie derivative.

$$\begin{aligned}\mathcal{L}_\xi g_{\mu\nu} =&\xi^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho}\\ =&\xi^\lambda \nabla_\lambda g_{\mu\nu} + (\nabla_\mu \xi^\rho) g_{\rho\nu} + (\nabla_\nu \xi^\rho) g_{\mu\rho}\\ =&0 + \nabla_\mu(\xi^\rho g_{\rho\nu}) + \nabla_\nu(\xi^\rho g_{\mu\rho})\\ =&\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 2\nabla_{(\mu}\xi_{\nu)}\end{aligned}$$

The note that the derivative is covariant.

def. We define the Killing equation as the following Lie derivative being vanishing.

$$0 = \mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

The equation is what we need to solve for a given metric (background) to obtain its symmetry. Then, each solution ξ^μ (which we call killing vectors) corresponds to a certain symmetry.

def. Noether current (conserved) is defined as the contraction of the energy-momentum tensor with the killing tensor.

$$J^\mu = T^\mu_\nu \xi^\nu$$

The following quantity has a tensor density of 0. Note that

$$\begin{aligned}\nabla_\mu J^\mu =&\nabla_\mu(T^\mu_\nu \xi^\nu)\\ =&(\nabla_\mu T^\mu_\nu)\xi^\nu + T^{\mu\nu}\nabla_{(\mu}\xi_{\nu)} = 0\\ \partial_\mu(\sqrt{-g}J^\mu) =&0\end{aligned}$$

We have used the Leibniz rule and the fact that $T^{\mu\nu} = T^{(\mu\nu)}$. As the divergence is zero, the current is conserved. We thus identify that there is a corresponding current that is conserved for each Killing vector.

rmk. If $g_{\mu\nu}$ is independent of a certain coordinate, e.g. $t = x^0$, $\xi^\mu \partial_\mu = \partial_t$ is a killing vector, and $\xi^\mu = (1, 0, 0, 0)$ is a constant vector. Vertifying is trivial.

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + 0 + 0 = \partial_t g_{\mu\nu} = 0$$

We can see how energy is given as

$$E = \int \sqrt{-g} J^0 \, dx^3 = \int \sqrt{-g} T^0_0 \, dx^3$$

while the Noether charge is given as

$$Q_\xi = \int \sqrt{-g} T^0_\nu \xi^\nu \, dx^3$$

recall. For a flat metric, $g_{\mu\nu} = \eta_{\mu\nu}$. We now try to solve the Killing equation.

$$\begin{aligned}
0 =& \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \\
0 =& \partial_\lambda \partial_\mu \xi_\nu + \partial_\lambda \partial_\nu \xi_\mu \\
=& \partial_\lambda \partial_\mu \xi_\nu + \partial_\nu \partial_\lambda \xi_\mu \\
=& \partial_\lambda \partial_\mu \xi_\nu - \partial_\nu \partial_\mu \xi_\lambda \\
\partial_\lambda \partial_\mu \xi_\nu =& - \partial_\lambda \partial_\nu \xi_\mu = \partial_\nu \partial_\mu \xi_\lambda \\
\partial_\mu \partial_\lambda \partial_\nu =& - \partial_\mu \partial_\nu \xi_\lambda = 0
\end{aligned}$$

thus, $0 = \partial_\mu \partial_\nu \xi_\lambda$ and ξ^μ must be linear in x^λ. We have

$$\begin{aligned}
\xi^\mu =& C^\mu_\nu x^\nu + C^\mu \\
\xi_\mu =& C_{\mu\nu} x^\nu + C_\mu \\
\partial_\lambda \xi_\mu =& C_{\mu\lambda}
\end{aligned}$$

The Killing equation becomes $C_{\mu\lambda} + C_{\lambda\mu} = 0$ and $C_{\mu\lambda} = C_{[\mu\lambda]} = -C_{\lambda\mu}$, and

$$\xi^\mu = C^\mu_\nu x^\nu + C^\mu$$

where $C^{\mu\nu} = -C^{\nu\mu}$ and C^μ are constant. The prior are the Lorentz symmetries while the latter are the translational symmetries, combining to become the Poincare symmetries.

recall. We return to the linearized Einstein field equations. $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$. The inverse metric satisfies $g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}$, where $h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma} h_{\rho\sigma}$. Then the Christoffel symbol became

$$\begin{aligned}
\Gamma^\lambda_{\mu\nu} =& \frac{1}{2}\eta^{\lambda\rho}(\partial_\mu h_{\rho\nu} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}) \\
=& \frac{1}{2}(\partial_\mu h^\lambda_\nu + \partial_\nu h^\lambda_\mu - \partial^\lambda h_{\mu\nu})
\end{aligned}$$

The curvature tensor approximately becomes

$$\begin{aligned}
R^\kappa_{\lambda\mu\nu} \approx& \partial_\mu \Gamma^\kappa_{\nu\lambda} - \partial_\nu \Gamma^\kappa_{\mu\lambda} \\
\approx& \frac{1}{2}\partial_\mu(\partial_\nu h^\kappa_\lambda + \partial_\lambda h^\kappa_\nu - \partial^\kappa h_{\lambda\nu}) - \frac{1}{2}\partial_\nu(\partial_\mu h^\kappa_\lambda + \partial_\lambda h^\kappa_\mu - \partial^\kappa h_{\lambda\mu}) \\
\approx& \frac{1}{2}\partial_\mu \partial_\lambda h^\kappa_\nu - \frac{1}{2}\partial_\mu \partial^\kappa h_{\lambda\nu} - \frac{1}{2}\partial_\nu \partial_\lambda h^\kappa_\mu + \frac{1}{2}\partial_\nu \partial^\kappa h_{\lambda\mu}
\end{aligned}$$

Contracting the above,

$$\begin{aligned} R_{\lambda\nu} =& \frac{1}{2}\partial_\lambda\partial_\mu h^\mu_\nu - \frac{1}{2}\Box h_{\lambda\nu} - \frac{1}{2}\partial_\nu\partial_\lambda h^\kappa_\kappa + \frac{1}{2}\partial_\nu\partial^\kappa h_{\kappa\lambda} \\ =& -\frac{1}{2}\Box h_{\lambda\nu} - \frac{1}{4}\partial_\nu(\partial_\lambda h^\kappa_\kappa - 2\partial^\kappa h_{\kappa\lambda}) - \frac{1}{4}\partial_\lambda(\partial_\nu h^\kappa_\kappa - 2\partial^\kappa h_{\kappa\nu}) \end{aligned}$$

lastly,

$$\begin{aligned} R =& -\frac{1}{2}\Box h^\lambda_\lambda - \frac{1}{4}\Box h^\kappa_\kappa + \frac{1}{2}\partial_\mu\partial_\nu h^{\mu\nu} - \frac{1}{4}\Box h^\kappa_\kappa + \frac{1}{2}\partial_\mu\partial_\nu h^{\mu\nu} \\ =& -\frac{1}{2}\Box h^\lambda_\lambda - \frac{1}{2}\Box h^\kappa_\kappa + \partial_\mu\partial_\nu h^{\mu\nu} \\ =& -\Box h^\lambda_\lambda + \partial_\mu\partial_\nu h^{\mu\nu} \end{aligned}$$

thm. Consider the diffeomorphism

$$\begin{cases} g_{\mu\nu} \to \delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} \\ g_{\mu\nu} \to g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu} \\ \eta_{\mu\nu} + h_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{L}_\xi(\eta + h)_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{L}_\xi\eta_{\mu\nu} + \mathcal{L}_\xi h_{\mu\nu} \end{cases}$$

Assume that ξ and h have the same orders of magnitude in order to ignore the last term. We arrive at

$$h_{\mu\nu} \to h_{\mu\nu} + \mathcal{L}_\xi\eta_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$$

def. The harmonic gauge is given as

$$\partial_\mu h^\nu_\nu - 2\partial_\nu h^\nu_\mu = 0$$

The gauge freedom is given as

$$\begin{aligned} h_{\mu\nu} \to& h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu \\ h^\nu_\nu \to& h^\nu_\nu + 2\partial_\mu\xi^\mu \\ \partial_\lambda h_{\mu\nu} \to& \partial_\lambda h_{\mu\nu} + \partial_\lambda\partial_\mu\xi_\nu + \partial_\lambda\partial_\nu\xi_\mu \\ \partial_\lambda h^\lambda_\mu \to& \partial_\lambda h^\lambda_\mu + \partial_\mu\partial_\nu\xi^\nu + \Box\xi \cdots \\ \partial_\mu h^\nu_\nu - 2\partial_\nu h^\nu_\mu \to& \partial_\mu h^\nu_\nu - 2\partial_\nu h^\nu_\mu - 2\Box\xi_\mu \end{aligned}$$

19 Lecture 19

recall. Last class, we have learnt the linearised Einstein equations by putting $g \approx \eta + h$ with $R_{\mu\nu} = -\Box h_{\mu\nu}/2$. We considered the following diffeomorphism

$$\delta h_{\mu\nu} = \mathcal{L}_\xi h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$$

resulting in

$$\delta(\partial_\lambda h^\lambda_\mu - \frac{1}{2}\partial_\mu h^\lambda_\lambda) = -2\Box\xi_\mu$$

we have the following transformation

$$h^{\text{old}}_{\mu\nu} \to h^{\text{new}}_{\mu\nu} = h^{\text{old}}_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$$

but

$$\partial_\lambda h^{\text{new}\lambda}_\mu - \frac{1}{2}\partial_\mu h^{\text{new}\lambda}_\lambda = 0 = \partial_\lambda h^{\text{old}\lambda}_\lambda - \frac{1}{2}\partial_\mu h^{\text{old}\lambda}_\lambda - \Box\xi_\mu$$

recall. We know the following facts from electrodynamics. If an equation is given like the following,

$$\Box\psi(x^0, \vec{x}) = 4\pi\rho(x^0, \vec{x})$$

the solution is given as

$$\psi(x^0, \vec{x}) = \int \frac{\rho(x^0 - |\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|}\, dx'^3$$

We previously saw that the following is equivalent to the field equations

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R =& 8\pi G T_{\mu\nu} \\ (1 - \frac{R}{2})R =& 8\pi G T^\lambda_\lambda \\ R =& \frac{16\pi G}{2 - D}T^\lambda_\lambda \end{aligned}$$

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} + \frac{g_{\mu\nu}}{2 - D}T^\lambda_\lambda)$$

$$\Box h_{\mu\nu} \approx -16\pi G(T_{\mu\nu} + \frac{\eta_{\mu\nu}}{2 - D}T^\lambda_\lambda)$$

we know that $h_{\mu\nu}$ is of the same order as $8\pi G T_{\mu\nu}$ as it is the variation from flat space. The above equation's solution is given as

$$h_{\mu\nu}(x^0, \vec{x}) = 4G\int \frac{T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^\lambda_\lambda}{|\vec{x} - \vec{x}'|}\, dx^3\Big|_{x^0_{\text{ret}} = x^0 - |\vec{x} - \vec{x}'|}$$

vertifying that it satisfies the gauge, we find

$$
\begin{aligned}
h^{\mu}_{\mu} =& 4G\int \frac{-T^{\lambda}_{\lambda}(x^0_{\text{ret}}, \vec{x}')}{|\vec{x}-\vec{x}'|}\ dx^3 \\
\partial_\lambda h^{\lambda}_{\mu} - \frac{1}{2}\partial_\mu h^{\lambda}_{\lambda} =& \partial_\lambda (h^{\lambda}_{\mu} - \frac{1}{2}\delta^{\lambda}_{\mu} h^{\nu}_{\nu}) \\
T^{\lambda}_{\mu} - \frac{1}{2}\delta^{\lambda}_{\mu} T^{\rho}_{\rho} - \frac{1}{2}\delta^{\lambda}_{\mu}(-T^{\rho}_{\rho}) =& T^{\lambda}_{\mu} \\
\partial_\lambda \int \frac{T^{\lambda}_{\mu}(x^0 - |\vec{x}-\vec{x}'|, \vec{x}')}{|\vec{x}-\vec{x}'|}\ dx^3 =& \int \frac{\partial_0 T^{0}_{\mu}(x^0 - |\vec{x}-\vec{x}'|, \vec{x}')}{|\vec{x}-\vec{x}'|} + \frac{\partial'_i T^{i}_{\mu}(x^0 - |\vec{x}-\vec{x}'|, \vec{x}')}{|\vec{x}-\vec{x}'|}\ dx^3 \\
&+ (\text{terms that vanish at infinity})
\end{aligned}
$$

20 Lecture 20

recall. Last class, we linearized the field equations into

$$
\Box h_{\mu\nu} = -\frac{16\pi G}{c^2}(T_{\mu\nu} + \frac{\eta_{\mu\nu}}{2-D}T^{\lambda}_{\lambda})
$$

and solved for the variation, obtaining

$$
h_{\mu\nu}(x^0, \vec{x}) = 4G\int \frac{T_{\mu\nu}(x^0 - |\vec{x}-\vec{x}'|, \vec{x}') - \frac{1}{2}\eta_{\mu\nu}T^{\lambda}_{\lambda}(x^0 - |\vec{x}-\vec{x}'|, \vec{x}')}{|\vec{x}-\vec{x}'|}\ d^3x'
$$

under the gauge

$$
2\partial_\lambda h^{\lambda}_{\mu} = \partial_\mu h^{\lambda}_{\lambda}
$$

Lets try and repeat the calculations. The Christoffel symbols became

$$
\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \approx \frac{1}{2}(\partial_\mu h^{\lambda}_{\nu} + \partial_\nu h^{\lambda}_{\mu} - \partial^{\lambda} h_{\mu\nu})
$$

Given the geodesic equation,

$$
0 = \ddot{x}^{\lambda} + \Gamma^{\lambda}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}
$$

we approximate it using the Christoffel symbol above to obtain

$$
0 \approx \ddot{x}^{\lambda} + (\partial_\mu h^{\lambda}_{\nu} - \frac{1}{2}\partial^{\lambda} h_{\mu\nu})\dot{x}^{\mu}\dot{x}^{\nu}
$$

rearranging,

$$
\ddot{x}^{\lambda} = \frac{1}{2}(\partial^{\lambda} h_{\mu\nu} - 2\partial_\mu h^{\lambda}_{\nu})\dot{x}^{\mu}\dot{x}^{\nu}
$$

For a non-relativistic/slow moving particle, we know that the front term is dominant, i.e., $\dot{x}^\mu = (\dot{x}^0, \dot{\vec{x}}) \approx (c, \vec{v}) \approx (c, 0)$, $|\vec{v}| << c$. Then, we get

$$\ddot{x}^\lambda \approx \frac{c^2}{2}\partial^\lambda h_{00} - c^2 \partial_0 h_0^\lambda \approx \frac{c^2}{2}\partial^\lambda h_{00} - c\frac{\partial}{\partial t} h_0^\lambda$$

and

$$\ddot{x}^\lambda \approx \frac{c^2}{2}\partial^\lambda h_{00}$$

implying

$$\ddot{\vec{x}} = \frac{c^2}{2}\nabla h_{00}$$

From the above, we try to find h_{00}, supposedly

$$h_{00} = \frac{4G}{c^2}\int \frac{T_{00} + \sum_{i=1}^3 T_{ii}}{|\vec{x} - \vec{x}'|}\, d^3x'$$

using

$$\begin{aligned} T_\lambda^\lambda =& T_0^0 + T_i^i = -T_{00} + T_i^i \\ T_{00} - \frac{1}{2}\eta_{00}T_\lambda^\lambda =& T_{00} + \frac{1}{2}(-T_{00} + T_i^i) = \frac{(T_{00} + \sum_{i=1}^3 T_{ii})(\vec{x}')}{2} \end{aligned}$$

Finally,

$$\ddot{\vec{x}} \approx G\nabla \int \frac{T_{00} + \sum_{i=1}^3 T_{ii}}{|\vec{x} - \vec{x}'|}\, d^3x'$$

In the Newtonian limit, with Newtonian gravity, we would expect

$$m\ddot{\vec{x}} = -\nabla V$$

and

$$\frac{V}{m} = -G\int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}\, d^3x'$$

thus we find the mass density to be

$$\rho(\vec{x}) = T_{00} + \sum_{i=1}^3 T_{ii}$$

thm. How does light behave under such pertubations of the metric? We know that light behaves like the following

$$\dot{x}^\mu \dot{x}^\nu (\eta_{\mu\nu} + h_{\mu\nu}) = 0$$

with

$$\dot{x}^\mu \dot{x}^\nu g_{\mu\nu} = \begin{cases} 0 & \text{light} \\ -1 & \text{massive particle} \\ 1 & \text{tachyon} \end{cases}$$

let $\dot{x}^\mu = (v, v, 0, 0)$ which would describe light moving along the x-direction or generally $\dot{x}^\mu = (v, v\hat{n})$, for a unit vector $\hat{n} \cdot \hat{n} = 1$. We state without derivation that when we plug this into the linearized field equations, we obtain the following metric. Consider the solar system test in a sphercal symmetric case. PPN stands for parametrised-post-newtonian analysis.

$$ds^2 = \Big(-1 + \frac{2GM}{r} + 2\beta_{\text{PPN}}\Big(\frac{GM}{r}\Big)^2 + \cdots\Big)dt^2 + \Big(1 + 2\gamma_{\text{PPN}}\frac{GM}{r} + \cdots\Big)d\vec{x} \cdot d\vec{x}$$

According to experiment, we know that $\beta_{\text{PPN}} \approx 1$ and $\gamma_{\text{PPN}} \approx 1$, which is exactly satisfied by the Einstein field equations with large precision.

rmk. Finding the average of the trace of the pertubtation,

$$\begin{aligned}\frac{1}{3}h_i^{\ i} =& \frac{1}{3}(T_i^{\ i} - \frac{3}{2}T_\lambda^{\ \lambda}) \\ =& \frac{1}{2}(T_{00} - \frac{1}{3}T_i^{\ i})\end{aligned}$$

we know that if $T_i^{\ i} \approx 0$ then $h_{00} \approx h_i^{\ i}$ and that $\gamma_{\text{PPN}} \approx 1$.

21 Lecture 21

recall. We calculate the Laplacian of the potential function like the following

$$\begin{aligned}&\nabla^2 \int \frac{\rho(x^0 - |\boldsymbol{x} - \boldsymbol{x}'|, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}\ d^3x' \\ =&\nabla \cdot \int \nabla\Big(\frac{\rho(x^0 - |\boldsymbol{x} - \boldsymbol{x}'|, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}\Big)\ d^3x' \\ =&\nabla \cdot \int \frac{\nabla\rho(x^0 - |\boldsymbol{x} - \boldsymbol{x}'|, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} + \rho(x^0 - |\boldsymbol{x} - \boldsymbol{x}'|, \boldsymbol{x}')\nabla\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|}\ d^3x' \\ =&\nabla \cdot \int \frac{\nabla(x^0 - |\boldsymbol{x} - \boldsymbol{x}'|)\partial_0\rho(x^0 - |\boldsymbol{x} - \boldsymbol{x}'|, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} - \frac{(\boldsymbol{x} - \boldsymbol{x}')\rho}{|\boldsymbol{x} - \boldsymbol{x}'|^3}\ d^3x' \\ =&-\nabla \cdot \int (\boldsymbol{x} - \boldsymbol{x}')\Big[\frac{\partial_0\rho(x^0 - |\boldsymbol{x} - \boldsymbol{x}'|, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|^2} + \frac{\rho(x^0 - |\boldsymbol{x} - \boldsymbol{x}'|, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}\Big]\ d^3x' \\ =&\nabla \cdot \int -\frac{(\boldsymbol{x} - \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|^2}\partial_0\rho(x^0 - |\boldsymbol{x} - \boldsymbol{x}'|, \boldsymbol{x}') + \rho(x^0 - |\boldsymbol{x} - \boldsymbol{x}'|, \boldsymbol{x}')\nabla\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|}\ d^3x' \\ =&-4\pi\rho + \partial_0^2 \int \frac{\rho}{|\boldsymbol{x} - \boldsymbol{x}'|}\ d^3x'\end{aligned}$$

Thus, we can conclude

$$\Box \int \frac{\rho(x^0 - |\boldsymbol{x} - \boldsymbol{x}'|, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}\ d^3x' = -4\pi\rho$$

thm. Today, we're going to obtain a spherically symmetric vaccum solution to the gravitational field equations.

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

As we learnt the Lie derivative, we know that such a solution should satisfy

$$\mathcal{L}_{L_a} g_{\mu\nu} = 0$$

for $a \in 1, 2, 3$. L_1, L_2, L_3 are nothing other than angular momenta, given by

$$L_3 = \frac{\partial}{\partial\phi} = L_3^\mu \partial_\mu = L_3^\phi \partial_\phi$$

and

$$\mathcal{L}_{L_a} T_{\mu\nu} = 0$$

when you impose all the following

$$\begin{cases} \mathcal{L}_{L_1} g_{\mu\nu} = 0 \\ \mathcal{L}_{L_2} g_{\mu\nu} = 0 \\ \mathcal{L}_{L_3} g_{\mu\nu} = 0 \end{cases}$$

note.

$$\mathcal{L}_v g_{\mu\nu} = v^\lambda g_{\mu\nu} + \partial_\mu v^\lambda g_{\lambda\nu} + \partial_\nu v^\lambda g_{\mu\lambda}$$

thm. You obtain the following form of proper displacement anzatz

$$ds^2 = A(t,r)dt^2 + 2B(t,r)dtdr + C(t,r)dr^2 + D(t,r)(d\theta^2 + \sin^2\theta d\phi^2)$$

We can reform the anzatz into

$$\begin{aligned} ds^2 =& \tilde{A}(t,r)[dt + d\alpha(t,r)]^2 + \beta(t,r)dr^2 + D(t,r)(d\theta^2 + \sin^2\theta d\phi^2) \\ =& \tilde{A}[d(t+\alpha)]^2 + \cdots \\ =& \tilde{A}dt_{\text{new}}^2 + \beta dr^2 + D(d\theta^2 + \sin\theta d\phi^2) \end{aligned}$$

where $t_{\text{new}} = t + \alpha(t,r)$. Expanding,

$$\begin{aligned} =& \tilde{A}(dt + dt\partial_r\alpha + dr\partial_r\alpha)^2 + \beta dr^2 \\ =& \tilde{A}((1+\partial_t\alpha)dt + \partial_r\alpha dr)^2 + \beta dr^2 \\ =& \tilde{A}(1+\partial_t\alpha)^2 dt^2 + 2\tilde{A}(1+\partial_t\alpha)\partial_r\alpha dtdr + [\tilde{A}(\partial_r\alpha)^2 + \beta]dr^2 \end{aligned}$$

this implies that

$$\begin{cases} A = \tilde{A}(1+\partial_t\alpha)^2 \\ B = \tilde{A}(1+\partial_t\alpha)\partial_r\alpha \\ C = \tilde{A}(\partial_r\alpha)^2 + \beta \end{cases}$$

Logic here is faulty, but we want to conclude that we don't need a cross term. We can also go into coordinate where $D = r_{\text{new}}^2$. In this new coordinate system, the proper distance becomes

$$ds^2 = A(t,r)dt^2 + B(t,r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

another popular choice is

$$ds^2 = C(t,\tilde{r})dt^2 + B(t,\tilde{r})[d\tilde{r}^2 + \tilde{r}^2(d\theta^2 + \sin^2\theta d\phi^2)]$$

the terms in the square brackets interestingly are spatically flat, and we call them isotropic coordinates in systems with spherical symmetry.