Frequency Response & Filters

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(6th Week Post-Experiment Lab Report)

I. EXPERIMENT DATA & ANALYSIS

V_{PP} (V)	time-constant (experimental) (µs)	time-constant (theoretical) (µs)	% error
3.88	890	1000	11.0

TABLE I: The RC circuit's time constant and percentage error.

Using Kirchhoff's law for DC circuits, we can construct the following formula.

$$\varepsilon = iR + \frac{q}{C} = R\frac{dq}{dt} + \frac{q}{C}$$

Solving the differential equation for the function $q(t)$ gives the following.

$$q(t) = C\varepsilon(1 - e^{-t/RC}) + q_0 e^{-t/RC}$$

The initial charge q_0 is generally zero. For the capacitor, the voltage across it is this charge function divided by the capacitance C, which gives the following.

$$V = \varepsilon(1 - e^{-t/RC})$$

As shown, the theoretical time constant, the time that the voltage value decreases from the peak-to-peak value to $1-e^{-1}$ times this peak-to-peak value, is RC, the product of the resistance and the capacitor.

$$10\ \text{k}\Omega \times 0.1\ \mu\text{F} = 1000\ \mu\text{s}$$

As the two figures accurately depict, the RC differentiator is the resistance's voltage, whereas the integrator is the capacitor's voltage. The Kirchhoff's law for circuits accurately explains this phenomenon.

$$V_{\text{out}} = \frac{1}{RC}\int V_{\text{in}}\, dt$$

The following equation is what is derived when the input voltage is taken as the voltage for the resistor and the output voltage is taken as the voltage for the capacitor.

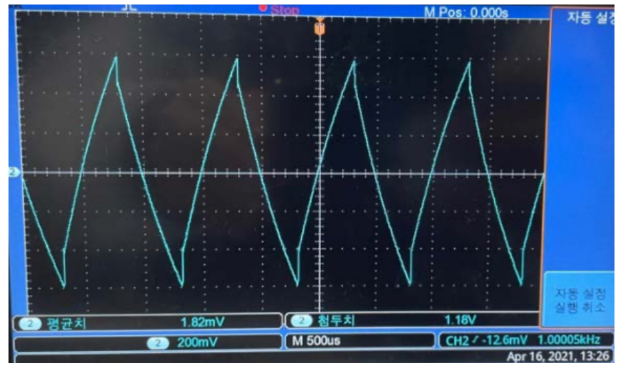

FIG. 1: A short period compared to the time constant inserted for the AC signal.

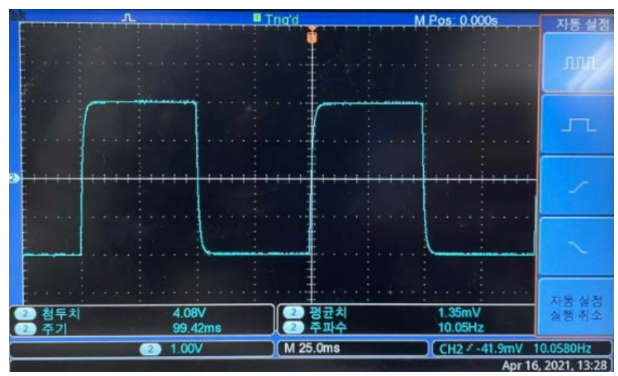

FIG. 2: A large period compared to the time constant inserted for the AC signal.

$$V_{\text{out}} = RC\frac{dV_{\text{in}}}{dt}$$

This equation is what is derived when the input voltage is taken as the voltage for the capacitor and the output voltage is taken as the voltage for the resistor.

For the second measurement set of the first experiment, the Bode plot for an RC circuit with a 1 kΩ resistor and a 1 µF capacitor. the result can be seen below. The third experiment was on the Bode plots for the RL circuit, where the same process of finding the time-constant, and Bode plots for the gain and phase were recorded. The results can be seen along with the RC circuit data as seen below.

The theory behind the Bode plot can be seen in the equation below, where the output voltage can be expressed in

cut-off frequency (experimental) (Hz)	cut-off frequency (theoretical) (Hz)	% error
166.0	159.2	4.277

TABLE II: The cut-off frequency for the RC circuit.

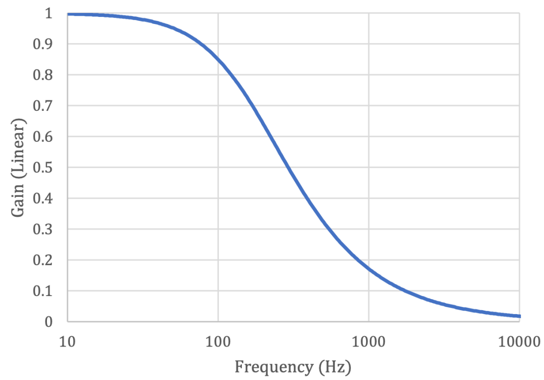

FIG. 3: RC circuit linear gain Bode plot.

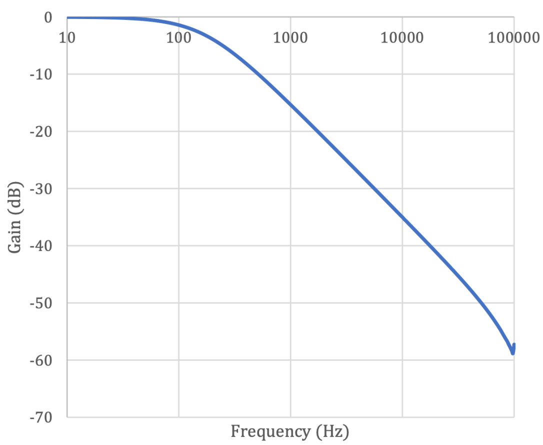

FIG. 4: RC circuit log gain Bode plot.

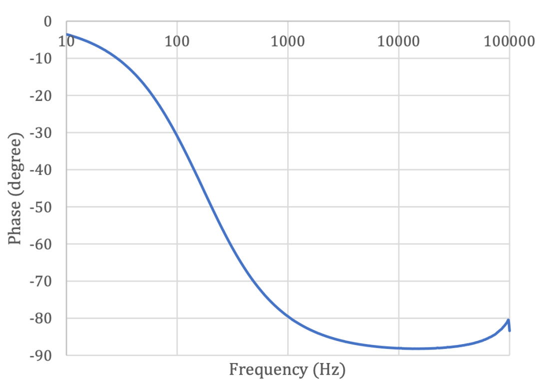

FIG. 5: RC circuit phase Bode plot.

V_{PP} (V)	time-constant (experimental) (µs)	time-constant (theoretical) (µs)	% error
4.00	8.60	10.00	14.00

TABLE III: The RL circuit's time constant and percentage error.

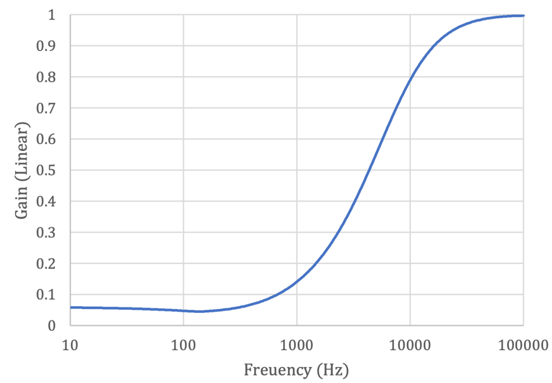

FIG. 6: RL circuit linear gain Bode plot.

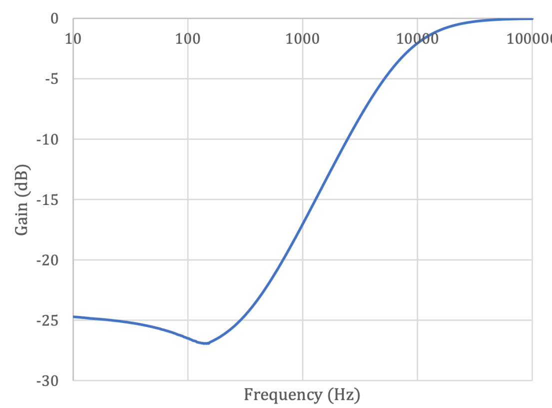

FIG. 7: RL circuit log gain Bode plot.

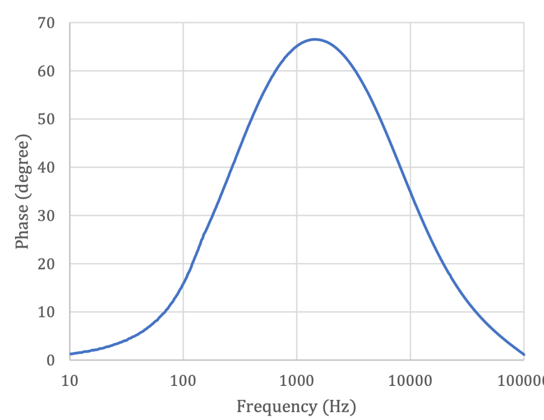

FIG. 8: RL circuit phase Bode plot.

terms of the input voltage expressed in terms of a cosine function. For the capacitor, the function becomes the following.

$$V_{\text{out}} = \frac{X_C}{R + X_C} V_{\text{in}} = \frac{\frac{1}{i\omega C}}{R + \frac{1}{i\omega C}} V_{\text{in}} = \frac{V_0}{i\omega RC + 1} \cos \omega t$$

At low frequencies, the impedance of a capacitor is high, so the magnitude of the response is relatively flat and close to zero dB. As the frequency increases, the impedance of the capacitor decreases inversely with frequency causing the magnitude response to decline. de. This decrease continues until the frequency approaches the resonant frequency where the impedance becomes significant and the magnitude response reaches its lowest point. At low frequencies, the phase shift across the capacitor is close to zero degrees. As the frequency increases, the phase shift across the capacitor decreases, reaching a minimum of -90 degrees at the resonant frequency.

For the capacitor, the function becomes the following.

$$V_{\text{out}} = \frac{X_L}{R + X_L} V_{\text{in}} = \frac{i\omega L}{R + i\omega L} V_{\text{in}} = \frac{V_0}{1 + \frac{R}{i\omega L}} \cos \omega t$$

At low frequencies, the impedance of an inductor is low, so the magnitude of the response is relatively flat and close to zero dB. As the frequency increases, the impedance of the inductor increases proportionally with frequency causing the magnitude response to rise. This increase continues until the frequency approaches the resonant frequency where the impedance becomes significant and the magnitude response reaches its peak. At low frequencies, the phase shift across the inductor is close to zero degrees. As the frequency increases, the phase shift across the inductor increases, reaching a maximum of +90 degrees at the resonant frequency.

A. RLC Circuits and Frequency Responses

Lets first investigate the general solution for the damped oscillator. The equation can be expressed as a second order ODE like the following, where appropriate substitutions can be made for $m = L$, $b = R$, and $k = 1/C$.

$$m\ddot{x} + b\dot{x} + kx = 0$$

dividing by m,

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{b}{k}x = 0$$

Assume an exponential solution

$$x(t) = e^{\gamma t} \qquad \dot{x}(t) = \gamma e^{\gamma t} \qquad \ddot{x}(t) = \gamma^2 e^{\gamma t}$$

Via substitution, the second order O.D.E. reduces to a quadratic equation

$$\gamma^2 e^{\gamma} + \frac{b}{m}\gamma e^{\gamma} t + \frac{k}{m} e^{\gamma} = 0$$

$$e^{\gamma}\left(\gamma^2 + \frac{b}{m}\gamma t + \frac{k}{m}\right) = 0$$

solving for γ,

$$\gamma_{1,2} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \left(\frac{k}{m}\right)}$$

Substituting $\omega_0 = \sqrt{k/m}$,

$$\gamma_{1,2} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \omega_0^2}$$

A underdamped cased of an oscillator is modelled through the case where the discriminant of the solution for γ above is negative ($\Delta < 0$). This results in imaginary solutions in finding γ. As mentioned in class, the general solution is linear combination of the two general solutions with the two integration constants multiplied to each exponential function. Namely,

$$x(t) = x_1 e^{\gamma_1 t} + x_2 e^{\gamma_2 t}$$

When the discriminant, as mentioned above, complies $(b/2m)^2 - \omega_0^2 < 0$, the complex number $i = \sqrt{i}$ can be substituted to the equation to make the equation the following form.

$$x(t) = x_1 e^{\gamma_1 t} + x_2 e^{\gamma_2 t}$$

$$x(t) = x_1 \exp\left(-\frac{b}{2m}t - \sqrt{\left(\frac{b}{2m}t\right)^2 - \omega_0^2 t}\right) + x_2 \exp\left(-\frac{b}{2m}t + \sqrt{\left(\frac{b}{2m}\right)^2 - \omega_0^2 t}\right)$$

$$x(t) = x_1 \exp\left(-\frac{b}{2m}t - i\sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2 t}\right) + x_2 \exp\left(-\frac{b}{2m}t + i\sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2 t}\right)$$

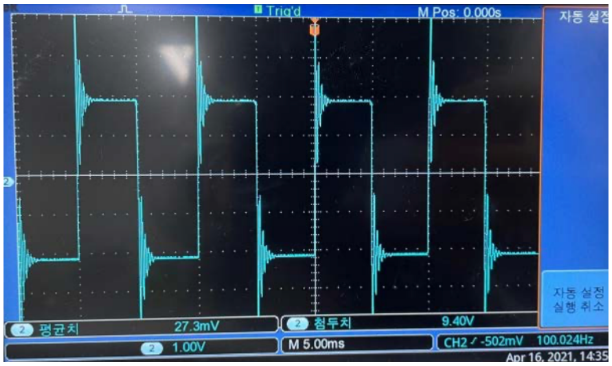

FIG. 9: RLC Circuit, under-damping.

FIG. 10: RLC Circuit, over-damping.

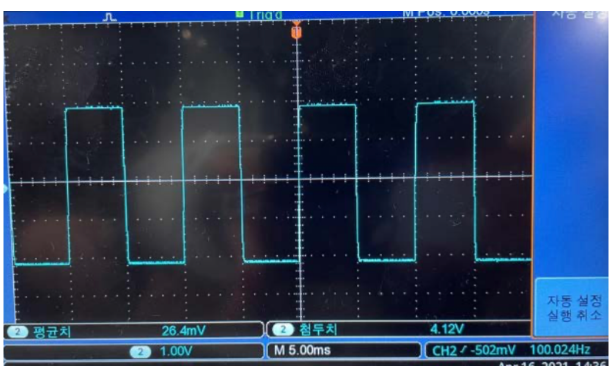

FIG. 11: RLC Circuit, criticial-damping.

resonance resistance (experimental) (kΩ)	resonance resistance (theoretical) (kΩ)	% error
0.7677	0.9381	18.16

TABLE IV: The cut-off frequency for the RC circuit.

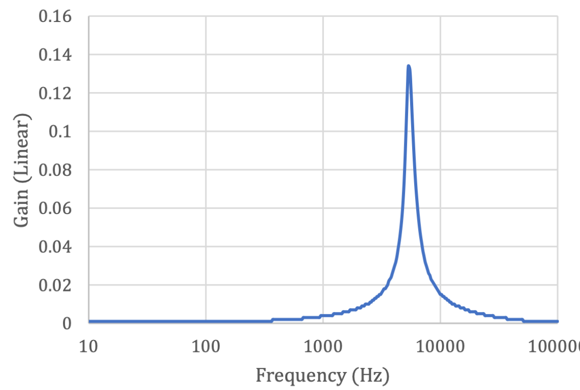

FIG. 12: RLC Circuit linear gain Bode plot.

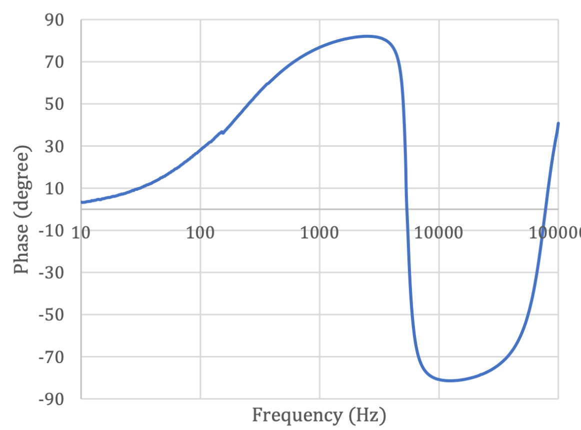

FIG. 13: RLC Circuit log gain Bode plot.

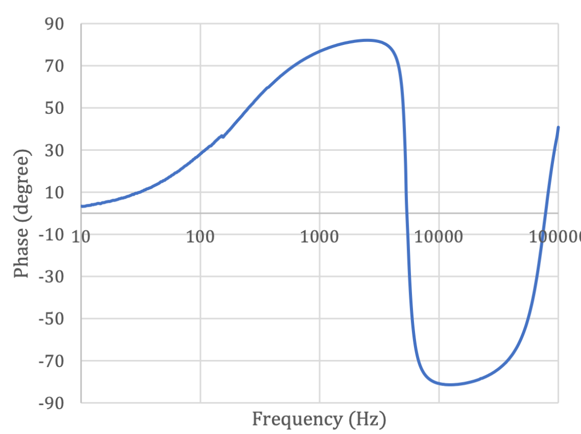

FIG. 14: RLC Circuit phase Bode plot.

resonance frequency (experimental) (kHz)	resonance frequency (theoretical) (kHz)	% error	phase (∘)
5.33	5.03	5.92	6.08

TABLE V: The resonance frequency of the RLC circuit.

bandwidth frequency (experimental) (kHz)	bandwidth frequency (theoretical) (kHz)	% error
876.5	81.2	979.9

TABLE VI: The bandwidth frequency of the RLC circuit.

Q-factor (experimental)	Q-factor (theoretical)	% error
6.08	62.01	90.19

TABLE VII: The Q-factor of the RLC circuit.

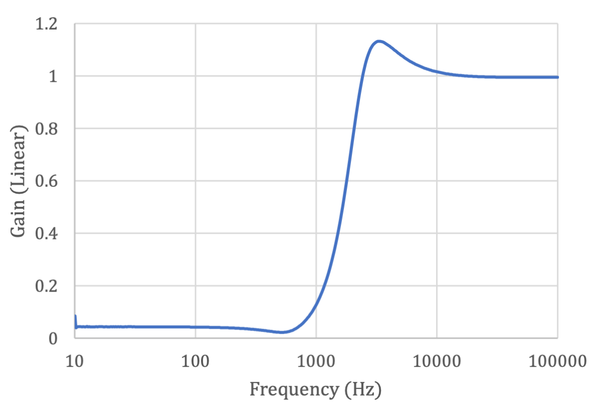

FIG. 15: T-filter linear gain Bode plot.

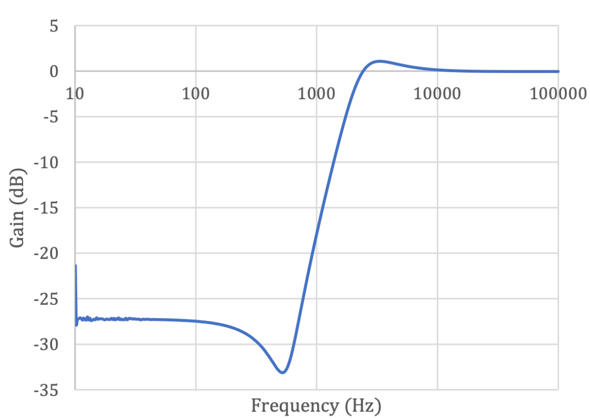

FIG. 16: T-filter log gain Bode plot.

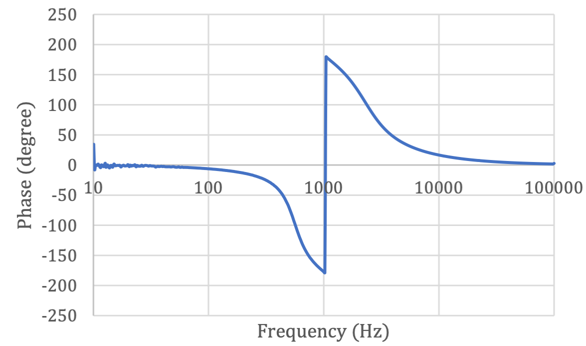

FIG. 17: T-filter phase Bode plot.

B. RLC Circuits and Resonance

C. Passive Filters (T-filters)

II. DISCUSSION

A. Goals and Recapitulation of Experiments

For the whole experiment, there were three sub-experiments, aimed at obtaining a total of five goals. The five goals were:

a. Observing voltage variations occurring in the transient and steady states of RC and RL circuits to understand the characteristics of resistors, capacitors, and inductors (RC and RL are each circuits created by capacitors and inductors, and the way that they act in transient and steady states occur due to the way that they store energy in their respective fields).

b. Observing underdamping, critical damping, and overdamping in an RLC circuit to gain understanding of the characteristics of the RLC circuit.

c. Understanding how the characteristics of resistors, capacitors, and inductors vary with frequency.

d. Through measuring frequency response curves using RC and RLC circuits, understanding phase difference and resonance phenomena.

e. Understanding the characteristics of passive filters using RLC circuits.

Consequently, in total, there were 4 different sets of measurements made for the first experiment, 1 set of measurement made for the second experiment, and 1 set of measurement made for the third experiment. The experiments can be seen in the list below.

1. The first measurement (set) of the first experiment was on constructing a RC circuit and observing the half-life of the voltage signal. Afterwards, the time constant of the circuit was calculated and compared with the theoretical value. Sine waves of large and short periods

compared to this time constant was inserted to see how the wave signals would change in shape.

2. The second measurement (set) of the first experiment was on creating the bode plot for the same RC circuit, but with different resistance and capacitor values. The point where the voltage gain became $1/\sqrt{2}$ was found and the point's frequency and phase was measured. This particular measurement was the measured cut-off frequency (or half-power frequency). This value was also measured with a theoretical value.

3. The third measurement (set) of the first experiment was on doing the same process denoted above for a RL circuit (time constant and bode plotting).

4. The fourth measurement (set) of the first experiment was on creating a RLC circuit and observing under-damping, critical-damping, and over-damping using the oscilloscope. The critical resistance was measured and compared with a theoretical value.

5. The first measurement (set) of the second experiment was on creating a bode plot for a RLC circuit and recording the bandwidth, Q-factor, and resonance frequency. All these values were compared with a theoretical value.

6. The first measurement (set) of the third experiment was on constructing a T-filter and creating a bode plot for a load (resistor).

B. Evaluation and Error Assessment

Throughout the experiment, the errors were minimal, showing a successful verification of our theory. However, there were particularly high deviations in the second experiment, where there existed extremely high errors in the bandwidth frequency and the Q-factor. Some sources of these errors can be summed up like the following.

Load effects introduced by the measurement equipment itself. When connecting instruments such as oscilloscopes or function generators to the circuit, their input impedance might interact with the circuit impedance, altering its behavior. For instance, if the input impedance of the measuring device is not properly matched with the impedance of the circuit, it can lead to reflections, impedance mismatches, and changes in signal amplitudes. These effects can distort the readings, making it challenging to accurately determine the bandwidth frequency and Q-factor. To mitigate these load effects, careful consideration should be given to the impedance matching between the circuit and the measurement equipment. Techniques such as using buffer amplifiers or impedance matching networks can help minimize these effects and ensure more accurate measurements.

Parasitic capacitance or inductance. Parasitic capacitance refers to unintended capacitance that exists between conductors in the circuit, while parasitic inductance refers to unintended inductance. These parasitic elements can affect the resonance characteristics of the circuit, leading to deviations from theoretical expectations.

Variations in temperature, humidity, and electromagnetic interference (EMI) can all influence the behavior of electronic components and the performance of the circuit. Temperature fluctuations can affect the characteristics of components such as resistors, capacitors, and inductors, leading to changes in their values and thus altering the resonance frequency and Q-factor of the circuit. Similarly, humidity levels can impact the dielectric properties of capacitors and affect their performance.

[1] Halliday, D., Resnick, R., and Krane, K. Physics, volume 2. *Wiley, 2nd Edition* (2010), pp. 845–853.
[2] The Sogang University Physics Department. Experimental physics 1 manual. *"Frequency Responce, Filters"*.
[3] Wikipedia. Bode plot. `https://en.wikipedia.org/wiki/Bode_plot`.
[4] Wikipedia. Q factor. `https://en.wikipedia.org/wiki/Q_factor`.
[5] Wikipedia. Rlc circuit. `https://en.wikipedia.org/wiki/RLC_circuit`.